Differential geometry of orbit space of extended Jacobi group A_n

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SISSA, Trieste

August 15, 2019

Dubrovin-Frobenius Manifolds

A Dubrovin-Frobenius structure on the manifold M is the data (M, $\bullet$, $<,>$, e, E) satisfying:

1. $\eta := <,>$ is a flat pseudo-Riemannian metric;
2. $\bullet$ is product of Frobenius algebra on $T_m M$ which depends smoothly on m;
3. e is the unity vector field for the product $\bullet$ and $\nabla e = 0$;
4. $\nabla_w c(x, y, z)$ is symmetric, where $c(x, y, z) := < x \bullet y, z >$;
5. A linear vector field $E \in \Gamma(M)$ must be fixed on M, i.e. $\nabla\nabla E = 0$ such that:

$$L_E <,> = (2-d) <,>,$$
$$L_E \bullet = \bullet,$$
$$L_E e = e.$$

The function $F(t), t = (t^1, t^2, .., t^n)$ is a solution of WDVV equation if its third derivatives

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \tag{1}$$

satisfies the following conditions:

1. $$\eta_{\alpha\beta} = c_{1\alpha\beta}$$

   is constant nondegenerate matrix.
2. The function
   $$c_{\alpha\beta}^{\gamma} = \eta^{\gamma\delta} c_{\alpha\beta\delta}$$

   is structure constant of assosciative algebra.
3. F(t) must be quasihomogeneous function

   $$F(c^{d_1} t^1, .., c^{d_n} t^n) = c^{d_F} F(t^1, .., t^n)$$

   for any nonzero c and for some numbers $d_1, ..., d_n, d_F$.

Introduction

Theorem (Dubrovin 1992)

There is a one to one correspondence between a Dubrovin-Frobenius manifold and solutions of WDVV equation.

Intersection form and Monodromy

The intersection form is the bilinear pairing in T^*M defined by:

$$(\omega_1, \omega_2)^* := \iota_E(\omega_1 \bullet \omega_2)$$

where $\omega_1, \omega_2 \in T^*M$ and $\bullet$ is the induced Frobenius algebra product in the cotangent space. Let us denote by g^* the intersection form.

The intersection form g of a Dubrovin-Frobenius manifold is a flat almost everywhere nondegenerate metric. Let us define:

$$\Sigma = \{x \in M : det(g) = 0\}$$

Hence, the linear system of differential equations determining g^*-flat coordinates has poles, and consequently its solutions $x_a(t^1, .., t^n)$ are multivalued, where $(t^1, .., t^n)$ are flat coordinates of η. The analytical continuation of the solutions $x_a(t^1, .., t^n)$ has monodromy corresponding to loops around Σ. This gives rise to a monodromy representation of $\pi_1(M \setminus \Sigma)$, which is called Monodromy of the Dubrovin-Frobenius manifold.

Frobenius Manifolds as Ω/W

Theorem (Dubrovin Conjecture, Hertling 1999)

Any irreducible semisimple polynomial Dubrovin-Frobenius manifold with positive invariant degrees is isomorphic to the orbit space of a finite Coxeter group.

Main Point

Differential geometry of the orbit spaces of reflection groups and of their **extensions** $\mapsto$ Dubrovin-Frobenius manifolds.

Example: W is Extended affine Weyl Group [Dubrovin, Zhang 1998] and for Jacobi groups [Bertola 1999].

Hurwitz space as Frobenius manifold

The Hurwitz space $H_{g,n_0,n_1,..,n_m}$ is the moduli space of curves C_g of genus g endowed with $N = m + 1 + n_0 + ..n_m$ branched covering λ of $\mathbb{C}P^1$, $\lambda : C_g \mapsto \mathbb{C}P^1$ with $m + 1$ branching points over ∞ in $\mathbb{C}P^1$ of branching degree $n_j + 1$, $j = 0, .., m$.

Examples of Hurwitz spaces

Example 1:
For $H_{0,1}$:

1. $\lambda(p, v) = p^2 - v^2$;
2. Monodromy action: $v \mapsto -v$;
3. $H_{0,1} \cong \mathbb{C}/A_1$.

Example 2:
For $H_{0,0,0}$:

1. $\lambda(p, a, b) = p + \frac{a}{p-b}$;
2. Monodromy action: $(x_1, x_2) \mapsto (x_1 + m, -x_2 + n)$;
3. $H_{0,0,0} \cong \mathbb{C}^2/\tilde{A}_1$.

For $H_{1,1}$:

1. $\lambda(v, v_0, \phi, \tau) = e^{2\pi i\phi}\frac{\theta_1(v-v_0|\tau)\theta_1(v+v_0|\tau)}{\theta_1^2(v|\tau)}$;
2. Monodromy action:
3. $(\phi, v_0, \tau) \mapsto (\phi, -v_0, \tau)$
4. $(\phi, v_0, \tau) \mapsto (\phi - nv_0 - \frac{n^2}{2}, v_0 + m + n\tau, \tau)$;
5. $(\phi, v_0, \tau) \mapsto (\phi - \frac{cv_0^2}{c\tau+d}, \frac{v_0}{c\tau+d}, \frac{a\tau+b}{c\tau+d})$;

$$H_{1,1} \cong \mathbb{C}^3/\mathcal{J}(A_1).$$

Problem Setting

$H_{1,1} \cong \mathbb{C}^3/\mathcal{J}(A_1)$
Example of Orbit space of Jacobi Group

$H_{0,0,0} \cong \mathbb{C}^2/\tilde{A}_1$
Example of Orbit space of Extended Affine Weyl Group

Mixed of Extended Affine Weyl Group + Jacobi Group?

$$H_{1,0,0} \cong \mathbb{C}^4/W$$

Results

$$\begin{array}{ccc} H_{0,0,0} \cong \mathbb{C}^2/\tilde{A}_1 & \longleftarrow & H_{0,1} \cong \mathbb{C}/A_1 \\ \downarrow & & \downarrow \\ H_{1,0,0} \cong \mathbb{C}^4/\mathcal{J}(\tilde{A}_1) & \longleftarrow & H_{1,1} \cong \mathbb{C}^3/\mathcal{J}(A_1) \end{array}$$

1. $H_{0,1}$, g=0, 1 double pole.
2. $H_{0,0,0}$,g=0, 2 simple pole.
3. $H_{1,1}$, g=1, 1 double pole.
4. $H_{1,0,0}$, g=1, 2 simple pole.

Results

For $(\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H})/\mathcal{J}(\tilde{A}_1)$
$\mathcal{J}(\tilde{A}_1) \curvearrowright \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H}$

$$(\phi, v_0, v_2, \tau) \mapsto (\phi, -v_0 + 2m_0, v_2 + 2m_2, \tau)$$
$$(\phi, v_0, v_2, \tau) \mapsto$$
$$(\phi - 2(n_0 v_0 - n_2 v_2) + (n_0^2 - n_2^2)\tau, v_0 + m_0 + n_0\tau, v_2 + m_2 + n_2\tau, \tau)$$
$$(\phi, v_0, v_2, \tau) \mapsto (\phi - \frac{c(v_0^2 - v_2^2)}{c\tau + d}, \frac{v_0}{c\tau + d}, \frac{v_2}{c\tau + d}, \frac{a\tau + b}{c\tau + d}) \tag{2}$$

$$\begin{aligned}[(\phi, v_0, v_2, \tau)] &\leftrightarrow e^{2\pi i\phi}\frac{\theta_1(v - v_0|\tau)\theta_1(v + v_0|\tau)}{\theta_1(v - v_2|\tau)\theta_1(v + v_2|\tau)} \\ &= \varphi_0 + \varphi_1[\zeta(v - v_2|\tau) - \zeta(v + v_2|\tau) + 2\zeta(v_2|\tau)]\end{aligned} \tag{3}$$

The invariant functions of $\mathcal{J}(\tilde{A}_1)$ of weight k, and index m are functions on $\Omega = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H} \ni (\phi, v_0, v_2, \tau)$ holomorphic on (v_0, ϕ, τ), and meromorphic on v_2 which satisfy

$$
\begin{aligned}
&E\varphi(\phi, v_0, v_2, \tau) := -\frac{1}{2\pi i}\frac{\partial}{\partial \phi}\varphi(\phi, v_0, v_2, \tau) = m\varphi(\phi, v_0, v_2, \tau) \\
&\varphi(\phi, v_0, v_2, \tau) = \varphi(\phi, -v_0, v_2, \tau) \\
&\varphi(\phi, v_0, v_2, \tau) = \\
&\varphi(\phi - 2n_0 v_0 - n_0^2\tau + 2n_2 v_2 + n_2^2\tau, v_0 + m_0 + n_0\tau, v_2 + m_2 + n_2\tau, \tau) \\
&\varphi(\phi, v_0, v_2, \tau) = (c\tau + d)^{-k}\varphi(\phi - \frac{c(v_0^2 - v_2^2)}{2(c\tau + d)}, \frac{v_0}{c\tau + d}, \frac{v_2}{c\tau + d}, \frac{a\tau + b}{c\tau + d})
\end{aligned}
\tag{4}
$$

The space of Jacobi forms of weight k, and index m is denoted by $J_{k,m}^{\tilde{A}_1}$.

Results

For $H_{1,0,0}$:

1. $ds^2 = 2dv_0^2 - 2dv_2^2 + 2d\phi d\tau$
2. $e = \frac{\partial}{\partial\varphi_0}$;
3. $E = \varphi_0 \frac{\partial}{\partial\varphi_0} + \varphi_1 \frac{\partial}{\partial\varphi_1}$;
4. $L_e g^* = \eta^*$
5. $(t^1, t^2, t^3, t^4) = (\varphi_0 + 2\varphi_1 \frac{\theta_1'(v_2|\tau)}{\theta_1(v_2|\tau)}, \varphi_1, v_2, \tau)$
6. $F^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\lambda}\frac{\partial^2 F}{\partial t^\mu \partial t^\lambda} = \frac{g^{\alpha\beta}}{deg(g^{\alpha\beta})}$
7. $F(t^1, t^2, t^3, t^4) = \frac{i(t^1)^2 t^4}{4\pi} - 2t^1 t^2 t^3 + (t^2)^2 \cdot Log(\frac{\pi\Theta_1'(0|t^4)}{t^2\Theta_1(2w|t^4)})$.

Results

$$\begin{array}{ccc} H_{0,n-1,0} \cong \mathbb{C}^{n+1}/\tilde{A}_n & \longleftarrow & H_{0,n} \cong \mathbb{C}^{n}/A_n \\ \downarrow & & \downarrow \\ H_{1,n-1,0} \cong \mathbb{C}^{n+3}/\mathcal{J}(\tilde{A}_n) & \longleftarrow & H_{1,n} \cong \mathbb{C}^{n+2}/\mathcal{J}(A_n) \end{array}$$

1. $H_{0,n}$, g=0, 1 pole of order n;
2. $H_{0,n-1,0}$,g=0, 1 simple pole, 1 pole of order n-1;
3. $H_{1,n}$, g=1, 1 pole of order n;
4. $H_{1,n-1,0}$, g=1, 1 simple pole, 1 pole of order n-1.

Thank you!

Sketch of the proof

Sketch of the construction for $\mathcal{J}(\tilde{A}_n)$:
1^{st} Step: [Construction of the orbit space]
Consider the action $\mathcal{J}(\tilde{A}_n) \curvearrowright \Omega = \mathbb{C} \oplus \mathbb{C}^{n+1} \oplus \mathbb{H}$

Definition 1 (Jacobi group of $\tilde{A}_n$)

The "Jacobi group of $\tilde{A}_n$" is represented on the Tits cone $\Omega = \mathbb{C} \oplus \mathbb{C}^{n+1} \oplus \mathbb{H}$ by the definition of the action $w \in \tilde{A}_n$, $t = (\lambda, \mu) \in (\mathbb{Z} + \tau\mathbb{Z})^{n+1}$, $\gamma \in SL_2(\mathbb{Z})$ as :

1. $w(\phi, v, \tau) = (\phi, wv, \tau)$
2. $t(\phi, v, \tau) = (\phi - < \mu, v > -\frac{1}{2} < \mu, \mu > \tau, v + \lambda + \tau\mu, \tau)$
3. $\gamma(\phi, v, \tau) = (\phi - \frac{c}{2(c\tau+d)} < v, v > \tau, \frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d})$

Sketch of the proof

Definition 2 (Jacobi forms)

The invariant functions of $\mathcal{J}(\tilde{A}_n)$ of weight k and index m are holomorphic functions on the Tits cone $\Omega = \mathbb{C} \oplus \mathbb{C}^n \oplus \mathbb{H}$, and meromorphic in the last variable v_{n+1} such that

1. $E\varphi(\phi, v, \tau) := \frac{1}{2\pi i}\frac{\partial}{\partial \phi}\varphi(\phi, v, \tau) = m\varphi(\phi, v, \tau)$
2. $\varphi(\phi, v, \tau) = \varphi(\phi, wv, \tau)$
3. $\varphi(\phi, v, \tau) = \varphi(\phi - < t, v > -\tau < t, t >, v + \lambda + t\tau, \tau)$
4. $\varphi(\phi, v, \tau) = (c\tau + d)^{-k}\varphi(\phi + c\frac{<v,v>}{2(c\tau+d)}, \frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d})$

and are locally bounded function on x as $Im(\tau) \mapsto \infty$. I will denote the space of Jacobi form of $\tilde{A}_n$ as $J_{\tilde{A}_n}$.

Theorem 3

The generators $(\varphi_0, \varphi_1, .., \varphi_n)$ *of the Algebra* $J_{\tilde{A}_n}$ *are given by the generating function:*

$$\begin{aligned}\lambda(v) =& e^{2\pi i\phi}\frac{\prod_{i=0}^{n}\theta_1(v-v_i|\tau)}{\theta_1^n(v|\tau)\theta_1(v+(n+1)v_{n+1}|\tau)}\\ =&\varphi_n\wp(v|\tau)^{(n-2)}+\varphi_{n-1}\wp(v|\tau)^{(n-3)}+..+\varphi_2\wp(v|\tau)\\ &+\varphi_1[\zeta(v|\tau)-\zeta(v+(n+1)v_{n+1})+2\zeta(\frac{n+1}{2}v_{n+1})]+\varphi_0\end{aligned} \quad (5)$$

Using the orbifold charts of $\Omega/\mathcal{J}(\tilde{A}_n)$, it is possible to prove that there is an unique bilinear form that transforms as a modular form of weight 2 under the action of $SL_2(\mathbb{Z})$,i.e under $\tau \mapsto \frac{a\tau+b}{c\tau+d}$, $ds^2 \mapsto \frac{ds^2}{(c\tau+d)^2}$. This bilinear form is:

$$ds^2 = ds^2_{\tilde{A}_n} + 2d\tilde{\phi}d\tau \qquad (6)$$

1. The unit vector field and Euler vector field are given in terms of the invariants. Indeed:

$$e = \frac{\partial}{\partial \varphi_0} \tag{7}$$

$$E = \varphi_0 \frac{\partial}{\partial \varphi_0} + \varphi_1 \frac{\partial}{\partial \varphi_1} + \varphi_2 \frac{\partial}{\partial \varphi_2} + .. + \varphi_n \frac{\partial}{\partial \varphi_n} \tag{8}$$

2. The last step is just to prove that $(\Omega/\mathcal{J}(\tilde{A}_n), g, L_e g, e, E)$ has a flat pencil strcuture, and therefore, a Frobenius structure. To prove it, note that $(\Omega/\mathcal{J}(\tilde{A}_n), g, e, E)$ is isomorphic to $(H_{1,n-1,0}, g, e, E)$, therefore, $(\Omega/\mathcal{J}(\tilde{A}_n), g, L_e g, e, E)$ has a flat pencil structure because $(H_{1,n-1,0}, g, L_e g, e, E)$ has it.

Hurwitz space as Frobenius manifold

The covering space $\hat{H}_{g,n_0,n_1,..,n_m}$ is defined:

$$\hat{H}_{g,n_0,n_1,..,n_m} := \{(C_g, \lambda, w_0, .., w_m, \{a_1, b_1, ..a_g, b_g\})\}$$

Locally in a neighbourhood of a covering of the described type, the set of branch points $\{\lambda_1, ..., \lambda_n\}$ gives coordinates on the Hurwitz space $\hat{H}_{g;n_0,...,n_m}$.
To build a frobenius structure on $\hat{H}_{g;n_0,...,n_m}$ take $\partial_i := \frac{\partial}{\partial \lambda_i}$,

1. the multiplication as $\partial_i \bullet \partial_j = \delta_{ij}\partial_i$,
2. $e = \sum \partial_i$,
3. $E = \sum \lambda^i \partial_i$,
4. $\eta = \sum res_{P_i} \frac{\phi^2}{d\lambda}(d\lambda^i)^2$,

where ϕ are the primary differential.

Formulas for g and η

$$< \partial_a, \partial_b >= - \sum_{|\lambda|<\infty} res_{d\lambda=0} \frac{\partial_a(\lambda(p)dp)\partial_b(\lambda(p)dp)}{d\lambda(p)} \tag{9}$$

$$(\partial_a, \partial_b) = - \sum_{|\lambda|<\infty} res_{d\lambda=0} \frac{\partial_a(Log\lambda(p)dp)\partial_b(Log\lambda(p)dp)}{dLog\lambda(p)} \tag{10}$$

$$c(\partial_a, \partial_b, \partial_c) = - \sum_{|\lambda|<\infty} res_{d\lambda=0} \frac{\partial_a(\lambda(p)dp)\partial_b(\lambda(p)dp)\partial_c(\lambda(p)dp)}{d\lambda(p)} \tag{11}$$

Flat coordinates of η on Hurwitz space

Theorem (Dubrovin 1992)

The corresponding flat coordinates $t_A, A = 1, ..., N$ consist of the five parts:

1. $t^{i;\alpha} = res_{\infty_i} \lambda^{\frac{-1}{n_i+1}} pd\lambda \quad i=0,...m, \quad \alpha = 1, ..., n_i;$
2. $p^i = v.p \int_{\infty 0}^{\infty_i} dp \quad i=0,...m;$
3. $q^i = res_{\infty i} \lambda dp \quad i=0,...m;$
4. $\tau^i = \int_{bi} dp \quad i=1,...g;$
5. $s^i = \int_{ai} \lambda dp \quad i=1,...g.$

Formulas

$$\wp(z,\omega,\omega') = \frac{1}{z^2} + \sum_{m^2+n^2\neq 0} \frac{1}{(z+2m\omega+2n\omega')^2} + \frac{1}{(2m\omega+2n\omega')^2} \tag{12}$$

$$\frac{d\zeta}{dz} = -\wp \tag{13}$$

$$\frac{dLog\sigma}{dz} = \zeta \tag{14}$$

$$\eta = \zeta(\omega,\omega,\omega') \tag{15}$$

$$\Theta_1(v|\tau) = 2\sum_{n=0}^{\infty}(-1)^n exp(i\pi(n+\frac{1}{2})^2\tau)sin((2n+1)\pi v) \tag{16}$$

$$\sigma(z,\omega,\omega') = 2\omega\frac{\Theta_1(\frac{z}{2\omega}|\tau)}{\Theta_1'(0|\tau)}exp(\frac{\eta z^2}{2\omega}) \tag{17}$$