# *The blocks of the periplectic Brauer algebra*

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Joint work with:
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# *Schur-Weyl duality*

$$GL(m) \longrightarrow (\mathbb{C}^m)^{\otimes n} \longleftarrow \mathbb{C}\Sigma_n$$

# *Schur-Weyl duality*

$$
\begin{array}{ccccc}
GL(m) & & & & \mathbb{C}\Sigma_n \\
& \searrow & & \swarrow & \\
\cup & & (\mathbb{C}^m)^{\otimes n} & & \cap \\
& \nearrow & & \nwarrow & \\
O(m) & & & & B_n(m)
\end{array}
$$

# *Schur-Weyl duality*

$$\begin{array}{ccccc} GL(2m) & \searrow & & \swarrow & \mathbb{C}\Sigma_n \\ \cup & & (\mathbb{C}^{2m})^{\otimes n} & & \cap \\ Sp(2m) & \nearrow & & \nwarrow & B_n(-2m) \end{array}$$

# *Schur-Weyl duality*

$$
\begin{array}{ccccc}
\mathfrak{gl}(m|2k) & \searrow & & \swarrow & \mathbb{C}\Sigma_n \\
\cup & & (\mathbb{C}^{m|2k})^{\otimes n} & & \cap \\
\mathfrak{osp}(m|2k) & \nearrow & & \nwarrow & B_n(m-2k)
\end{array}
$$

# *Schur-Weyl duality*

$$\begin{array}{ccccc} \mathfrak{gl}(m|m) & \searrow & & \swarrow & \mathbb{C}\Sigma_n \\ \cup & & (\mathbb{C}^{m|m})^{\otimes n} & & \cap \\ \mathfrak{pe}(m) & \nearrow & & \nwarrow & A_n \end{array}$$

# $(n, n)$-*Brauer diagrams*

$n$-northern nodes

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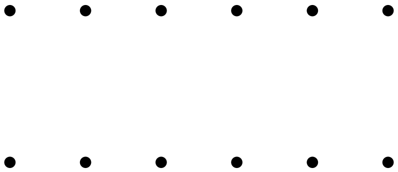
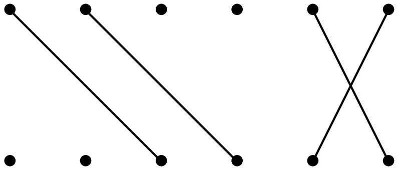


$n$-southern nodes

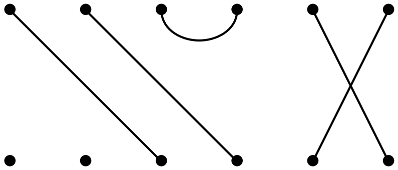
# $(n, n)$-*Brauer diagrams*

Propagating lines



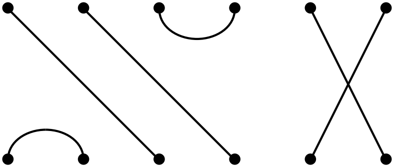
# $(n, n)$-*Brauer diagrams*

Propagating lines, cups



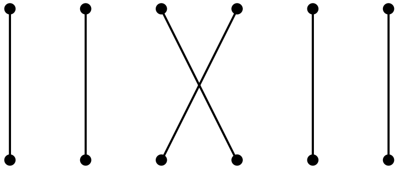
# $(n, n)$-*Brauer diagrams*

Propagating lines, cups and caps



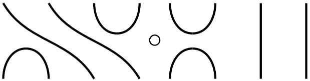
# $(n, n)$-Brauer diagrams

Only propagating lines $\Rightarrow$ Symmetric group



## *The (periplectic) Brauer algebra*

To multiply two Brauer diagrams:



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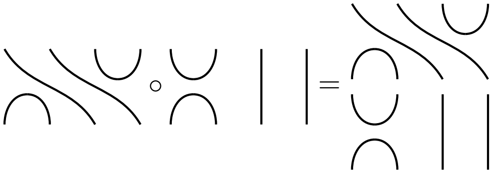

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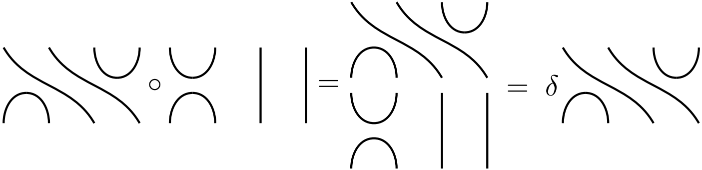


Replace each closed loop by $\delta$,
$\delta = 0$ for the periplectic case

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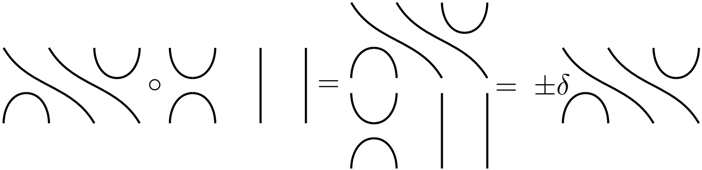


Calculate the appropriate sign using certain rules.

# *Labelling of simple modules*

*Theorem*

*The Brauer algebra $B_n(\delta)$, with $\delta \neq 0$:*

- *The p-restricted partitions of $n, n-2, n-4, \ldots, 0$ (n even)*
- *The p-restricted partitions of $n, n-2, n-4, \ldots, 1$ (n odd)*

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is $p$-restricted if $\lambda_i - \lambda_{i+1} < p$ for all $i$.

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# Labelling of simple modules

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*The Brauer algebra $B_n(0)$:*

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# *Labelling of simple modules*

*Theorem (Kujawa–Tharp 2017)*

*The periplectic Brauer algebra $A_n$:*

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# *Blocks*

$\lambda \sim \mu$ if there is a sequence

$$\lambda = \lambda_1, \lambda_2, \ldots, \lambda_t = \mu$$

with corresponding indecomposable $A$-modules

$$M_1, M_2, \ldots M_{t-1}$$

where $L(\lambda_i)$ and $L(\lambda_{i+1})$ appear as composition factors of $M_i$.

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Decomposition of $A$ in indecomposable two-sided ideals:

$$A = B_1 \oplus B_2 \oplus \cdots \oplus B_k,$$

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# Blocks of the Brauer algebra

**Theorem (Cox–De Visscher–Martin 2009)**

*The blocks of the Brauer algebra in characteristic zero correspond to orbits of the Weyl group of type D acting on partitions.*

**Theorem (King 2014)**

*The limiting blocks of the Brauer algebra in positive characteristic corresponds to orbits of the affine Weyl group of type D acting on partitions.*

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# *Blocks of the periplectic Brauer algebra*

***Theorem (Coulembier 2018)***

*In characteristic zero, two partitions belong to the same block iff they have the same 2-core.*

The 2-core: obtained by removing rim 2-hooks:



The possible 2-core:

$\rho_0 = \varnothing$, $\rho_1 =$ , $\rho_2 =$ , $\rho_3 =$ , $\ldots$

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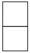
 or  .

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$$\ydiagram{2} \quad \text{or} \quad \ydiagram{1,1}\,.$$

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*If two partitions have the same 2-core, they belong to the same block.*

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*Consider the r-staircase partition $\rho_r$ with*

$$2r - 1 < p \quad \text{and} \quad \frac{r(r+1)}{2} + p - 2r > n.$$

*Then $\lambda \sim \rho_r$ if and only if the 2-core of $\lambda$ is $\rho_r$.*

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*If $\lambda$ has as 2-core $\rho_r$ not satisfying these conditions, then*

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*The block decomposition of $A_n$ is given by*

$$B_n(\kappa) \oplus \bigoplus_r B_n(\rho_r).$$

*Here $\kappa = (\square)$ if $n$ is odd or $\kappa = \emptyset$ if $n$ is even.*
*The sum is over all $r \geq 2$ such that*

- $2r - 1 < p$,
- $\frac{r(r+1)}{2} + p - 2r > n$,
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In particular if $n \geq (p^2 + 7)/8$, there is only one block.

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# BGG reciprocity

*Theorem*

*We have*

$$(P_n(\lambda) : W_n(\mu)) = [W_n(\mu^T) : L_n(\lambda^M)]$$

*where $\mu^T$ denotes the transpose of the partition $\mu$ and $\lambda^M$ denotes the Mullineux conjugate of the partition $\lambda$.*