Quantum Toroidal Superalgebras

Luan Bezerra
Joint work with Evgeny Mukhin

IUPUI

2019

Quantum Toroidal Superalgebra $\mathcal{E}_{m|n}(q_1, q_2, q_3)$.

1. Definition.
2. Vertex Representations.
3. Evaluation Homomorphism.

Quantum Toroidal Superalgebra $\mathcal{E}_{m|n}(q_1, q_2, q_3)$.

1. Definition.
2. Vertex Representations.
3. Evaluation Homomorphism.

Quantum Toroidal Superalgebra $\mathcal{E}_{m|n}(q_1, q_2, q_3)$.

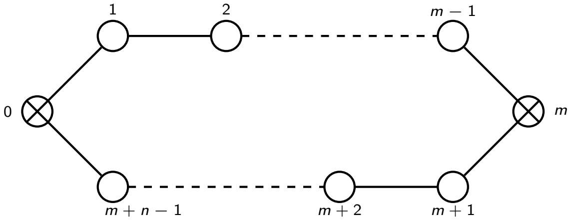
1. Definition.
2. Vertex Representations.
3. Evaluation Homomorphism.

Quantum Toroidal Superalgebra $\mathcal{E}_{m|n}(q_1, q_2, q_3)$.

- It is an affinization of $U_q\,\widehat{\mathfrak{gl}}_{m|n}$ with $m, n \geq 1$ and $m \neq n$;

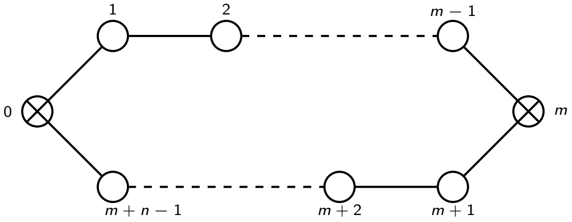
Quantum Toroidal Superalgebra $\mathcal{E}_{m|n}(q_1, q_2, q_3)$.

- It is an affinization of $U_q\,\widehat{\mathfrak{gl}}_{m|n}$ with $m, n \geq 1$ and $m \neq n$;
- Generators: $E_i(z), F_i(z), K_i^{\pm}(z), C,\ i \in \hat{I}$;



Quantum Toroidal Superalgebra $\mathcal{E}_{m|n}(q_1, q_2, q_3)$.

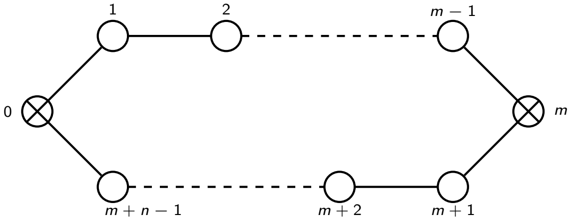
- It is an affinization of $U_q\,\widehat{\mathfrak{gl}}_{m|n}$ with $m, n \geq 1$ and $m \neq n$;
- Generators: $E_i(z), F_i(z), K_i^{\pm}(z), C,\ i \in \hat{I}$;



- Depends on two complex parameters: $q_1 q_2 q_3 = 1,\ q_2 = q^2$;

Quantum Toroidal Superalgebra $\mathcal{E}_{m|n}(q_1, q_2, q_3)$.

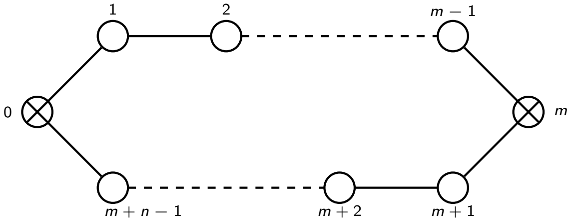
- It is an affinization of $U_q \widehat{\mathfrak{gl}}_{m|n}$ with $m, n \geq 1$ and $m \neq n$;
- Generators: $E_i(z), F_i(z), K_i^{\pm}(z), C, \ i \in \hat{I}$;



- Depends on two complex parameters: $q_1 q_2 q_3 = 1, \ q_2 = q^2$;
- It has a vertical subalgebra $U_q^{ver} \widehat{\mathfrak{sl}}_{m|n} \cong U_q \widehat{\mathfrak{sl}}_{m|n}$ given in new Drinfeld realization,

Quantum Toroidal Superalgebra $\mathcal{E}_{m|n}(q_1, q_2, q_3)$.

- It is an affinization of $U_q\,\widehat{\mathfrak{gl}}_{m|n}$ with $m, n \geq 1$ and $m \neq n$;
- Generators: $E_i(z), F_i(z), K_i^{\pm}(z), C,\ i \in \hat{I}$;



- Depends on two complex parameters: $q_1 q_2 q_3 = 1,\ q_2 = q^2$;
- It has a vertical subalgebra $U_q^{ver}\widehat{\mathfrak{sl}}_{m|n} \cong U_q\,\widehat{\mathfrak{sl}}_{m|n}$ given in new Drinfeld realization,
- and a horizontal subalgebra $U_q^{hor}\widehat{\mathfrak{sl}}_{m|n} \cong U_q\,\widehat{\mathfrak{sl}}_{m|n}$ given in Drinfeld-Jimbo realization.

Relations

$$K_i K_j = K_j K_i \,, \quad K_i E_j(z) K_i^{-1} = q^{A_{i,j}} E_j(z) \,, \quad K_i^{\pm}(z) K_j^{\pm}(w) = K_j^{\pm}(w) K_i^{\pm}(z) \,,$$

$$K_i^{\pm}(z) K_j^{\pm}(w) = K_j^{\pm}(w) K_i^{\pm}(z) \,,$$

$$\frac{d^{M_{i,j}} C^{-1} z - q^{A_{i,j}} w}{d^{M_{i,j}} C z - q^{A_{i,j}} w} K_i^{-}(z) K_j^{+}(w) = \frac{d^{M_{i,j}} q^{A_{i,j}} C^{-1} z - w}{d^{M_{i,j}} q^{A_{i,j}} C z - w} K_j^{+}(w) K_i^{-}(z) \,,$$

$$(d^{M_{i,j}} z - q^{A_{i,j}} w) K_i^{\pm}(C^{-\frac{1\pm 1}{2}} z) E_j(w) = (d^{M_{i,j}} q^{A_{i,j}} z - w) E_j(w) K_i^{\pm}(C^{-\frac{1\pm 1}{2}} z) \,,$$

$$[E_i(z), F_j(w)] = \frac{\delta_{i,j}}{q - q^{-1}} (\delta\left(C\frac{w}{z}\right) K_i^{+}(w) - \delta\left(C\frac{z}{w}\right) K_i^{-}(z)) \,,$$

$$[E_i(z), E_j(w)] = 0 \,, [F_i(z), F_j(w)] = 0 \qquad (A_{i,j}{=}0),$$

$$(d^{M_{i,j}} z - q^{A_{i,j}} w) E_i(z) E_j(w) = (-1)^{|i||j|} (d^{M_{i,j}} q^{A_{i,j}} z - w) E_j(w) E_i(z) \qquad (A_{i,j}{\neq}0),$$

$+$ Serre relations.

Properties

- Diagram isomorphism

$$\sigma : \mathcal{E}_{m|n}(q_1, q_2, q_3) \to \mathcal{E}_{m|n}(q_3, q_2, q_1), \qquad \sigma(E_i(z)) = E_{m-i}(z);$$

Properties

- Diagram isomorphism

$$\sigma : \mathcal{E}_{m|n}(q_1, q_2, q_3) \to \mathcal{E}_{m|n}(q_3, q_2, q_1), \qquad \sigma(E_i(z)) = E_{m-i}(z);$$

- change of parity isomorphism

$$\tau : \mathcal{E}_{m|n}(q_1, q_2, q_3) \to \mathcal{E}_{n|m}(q_3^{-1}, q_2^{-1}, q_1^{-1}), \quad \tau(E_i(z)) = E_{-i}(z);$$

Properties

- Diagram isomorphism

$$\sigma : \mathcal{E}_{m|n}(q_1, q_2, q_3) \to \mathcal{E}_{m|n}(q_3, q_2, q_1), \qquad \sigma(E_i(z)) = E_{m-i}(z);$$

- change of parity isomorphism

$$\tau : \mathcal{E}_{m|n}(q_1, q_2, q_3) \to \mathcal{E}_{n|m}(q_3^{-1}, q_2^{-1}, q_1^{-1}), \quad \tau(E_i(z)) = E_{-i}(z);$$

- it has a topological Hopf superalgebra structure:

$$\Delta E_i(z) = E_i(z) \otimes 1 + K_i^-(z) \otimes E_i(C \otimes z);$$

Properties

- Diagram isomorphism

$$\sigma : \mathcal{E}_{m|n}(q_1, q_2, q_3) \to \mathcal{E}_{m|n}(q_3, q_2, q_1), \qquad \sigma(E_i(z)) = E_{m-i}(z);$$

- change of parity isomorphism

$$\tau : \mathcal{E}_{m|n}(q_1, q_2, q_3) \to \mathcal{E}_{n|m}(q_3^{-1}, q_2^{-1}, q_1^{-1}), \quad \tau(E_i(z)) = E_{-i}(z);$$

- it has a topological Hopf superalgebra structure:

$$\Delta E_i(z) = E_i(z) \otimes 1 + K_i^-(z) \otimes E_i(C \otimes z);$$

- it has a two dimensional center.

Properties

- Different parity choices yield isomorphic superalgebras;

Properties

- Different parity choices yield isomorphic superalgebras;

- Miki automorphism: interchanges the vertical and horizontal subalgebras.

Level 1 Modules - Vertex Operators

- The Frenkel-Kac construction of $U_q\,\widehat{\mathfrak{gl}}_n$-modules was extended to the quantum toroidal (purely even) case [S]. We use the same technique to extend the construction of level 1 $U_q\,\widehat{\mathfrak{gl}}_{m|n}$-modules [KSU] to $\mathcal{E}_{m|n}$-modules.

Level 1 Modules - Vertex Operators

- The Frenkel-Kac construction of $U_q\,\widehat{\mathfrak{gl}}_n$-modules was extended to the quantum toroidal (purely even) case [S]. We use the same technique to extend the construction of level 1 $U_q\,\widehat{\mathfrak{gl}}_{m|n}$-modules [KSU] to $\mathcal{E}_{m|n}$-modules.
- As in the affine case, the modules obtained are not irreducible.

Level 1 Modules - Vertex Operators

- The Frenkel-Kac construction of $U_q\,\widehat{\mathfrak{gl}}_n$-modules was extended to the quantum toroidal (purely even) case [S]. We use the same technique to extend the construction of level 1 $U_q\,\widehat{\mathfrak{gl}}_{m|n}$-modules [KSU] to $\mathcal{E}_{m|n}$-modules.
- As in the affine case, the modules obtained are not irreducible.
- It is conjectured the irreducible modules are the kernel (or cokernel) of some screening operators.

Evaluation Map

Theorem

Fix $u \in \mathbb{C}^\times$. We have a surjective homomorphism of superalgebras $\mathrm{ev}_u : \mathcal{E}_{m|n} \to \widetilde{U}_q\, \widehat{\mathfrak{gl}}_{m|n}$ with $c^2 = q_3^{m-n}$. In particular, any admissible $U_q\widehat{\mathfrak{gl}}_{m|n}$-module on which the central element c acts as an arbitrary scalar α can be lifted to an $\mathcal{E}_{m|n}$-module choosing q_3 satisfying $\alpha^2 = q_3^{m-n}$.

Evaluation Map

Theorem

Fix $u \in \mathbb{C}^\times$. We have a surjective homomorphism of superalgebras $\mathrm{ev}_u : \mathcal{E}_{m|n} \to \widetilde{U}_q\,\widehat{\mathfrak{gl}}_{m|n}$ with $c^2 = q_3^{m-n}$. In particular, any admissible $U_q\widehat{\mathfrak{gl}}_{m|n}$-module on which the central element c acts as an arbitrary scalar α can be lifted to an $\mathcal{E}_{m|n}$-module choosing q_3 satisfying $\alpha^2 = q_3^{m-n}$.

- Generators with index $i \in I$ ($U_q^{ver}\widehat{\mathfrak{sl}}_{m|n}$) are mapped to generators;

Evaluation Map

Theorem

Fix $u \in \mathbb{C}^\times$. We have a surjective homomorphism of superalgebras $\mathrm{ev}_u : \mathcal{E}_{m|n} \to \widetilde{U}_q\, \widehat{\mathfrak{gl}}_{m|n}$ with $c^2 = q_3^{m-n}$. In particular, any admissible $U_q\widehat{\mathfrak{gl}}_{m|n}$-module on which the central element c acts as an arbitrary scalar α can be lifted to an $\mathcal{E}_{m|n}$-module choosing q_3 satisfying $\alpha^2 = q_3^{m-n}$.

- Generators with index $i \in I$ ($U_q^{ver}\widehat{\mathfrak{sl}}_{m|n}$) are mapped to generators;
- Generators corresponding to the node 0 are mapped to "dressed" currents:

Evaluation Map

$$E_0(z) \mapsto u^{-1} \qquad ,$$

Evaluation Map

$$E_0(z) \mapsto u^{-1} \qquad \mathsf{X}^-(z) \qquad ,$$

$$\mathsf{X}^-(z) = \left[\prod_{i=1}^{m+n-2}\left(1-\frac{z_{i+1}}{z_i}\right)\right] x_1^-(q^{-1}c^{-1}z_1)\cdots x_m^-(q^{-m}c^{-1}z_m)\times$$
$$\times\cdots x_{m+i}^-(q^{-m+i}c^{-1}z_{m+i})\cdots x_{m+n-1}^-(q^{-m+n-1}c^{-1}z_{m+n-1})\Big|_{z_1=\cdots=z_{m+n-1}=z},$$

$x_i^-(z)$ are current generators of $U_q\widehat{\mathfrak{gl}}_{m|n}$.

Evaluation Map

$$E_0(z) \mapsto u^{-1} \qquad\qquad \mathsf{X}^-(z) \qquad\qquad \mathcal{K},$$

$\mathcal{K}$ is in the weight lattice of $U_q\widehat{\mathfrak{gl}}_{m|n}$,

$$\mathsf{X}^-(z) = \left[\prod_{i=1}^{m+n-2}\left(1-\frac{z_{i+1}}{z_i}\right)\right] x_1^-(q^{-1}c^{-1}z_1)\cdots x_m^-(q^{-m}c^{-1}z_m)\times$$
$$\times\cdots x_{m+i}^-(q^{-m+i}c^{-1}z_{m+i})\cdots x_{m+n-1}^-(q^{-m+n-1}c^{-1}z_{m+n-1})\Big|_{z_1=\cdots=z_{m+n-1}=z},$$

$x_i^-(z)$ are current generators of $U_q\widehat{\mathfrak{gl}}_{m|n}$.

Evaluation Map

$$E_0(z) \mapsto u^{-1}\exp\left(A^-(z)\right)\mathrm{X}^-(z)\exp\left(A^+(z)\right)\mathcal{K},$$

$\mathcal{K}$ is in the weight lattice of $U_q\widehat{\mathfrak{gl}}_{m|n}$,

$A^\pm(z) = \sum_{r>0} A_{\pm r} z^{\mp r}$,

$$\mathrm{X}^-(z) = \left[\prod_{i=1}^{m+n-2}\left(1-\frac{z_{i+1}}{z_i}\right)\right] x_1^-(q^{-1}c^{-1}z_1)\cdots x_m^-(q^{-m}c^{-1}z_m) \times$$
$$\times \cdots x_{m+i}^-(q^{-m+i}c^{-1}z_{m+i})\cdots x_{m+n-1}^-(q^{-m+n-1}c^{-1}z_{m+n-1})\Big|_{z_1=\cdots=z_{m+n-1}=z},$$

$x_i^-(z)$ are current generators of $U_q\widehat{\mathfrak{gl}}_{m|n}$.

Evaluation Map

$$E_0(z) \mapsto u^{-1}\exp\left(A^-(z)\right)\mathrm{X}^-(z)\exp\left(A^+(z)\right)\mathcal{K},$$

$\mathcal{K}$ is in the weight lattice of $U_q\widehat{\mathfrak{gl}}_{m|n}$,

$A^\pm(z) = \sum_{r>0} A_{\pm r} z^{\mp r}$,

$A_r = -\frac{q-q^{-1}}{c^r-c^{-r}}\left(\tilde{h}_{0,r} + \sum_{i=1}^{m}(c^2q^i)^r h_{i,r} + \sum_{j=m+1}^{m+n-1}(c^2q^{2m-j})^r h_{j,r}\right)$,

$$\mathrm{X}^-(z) = \left[\prod_{i=1}^{m+n-2}\left(1-\frac{z_{i+1}}{z_i}\right)\right] x_1^-(q^{-1}c^{-1}z_1)\cdots x_m^-(q^{-m}c^{-1}z_m)\times$$
$$\times\cdots x_{m+i}^-(q^{-m+i}c^{-1}z_{m+i})\cdots x_{m+n-1}^-(q^{-m+n-1}c^{-1}z_{m+n-1})\Big|_{z_1=\cdots=z_{m+n-1}=z},$$

$x_i^-(z)$ are current generators of $U_q\widehat{\mathfrak{gl}}_{m|n}$.

References

K. Kimura, J. Shiraishi, and J. Uchiyama, *A level-one representation of the quantum affine superalgebra* $U_q(\widehat{\mathfrak{sl}}(M+1|N+1))$, Comm. Math. Phys. **188** (1997), no. 2, 367—378

K. Miki, *Toroidal braid group action and an automorphism of toroidal algebra* $U_q(\mathfrak{sl}_{n+1,tor})$ $(n \geq 2)$, Lett. Math. Phys. **47** (1999), no. 4, 365–378

Y. Saito, *Quantum toroidal algebras and their vertex representations*, Publ. Res. Inst. Math. Sci. **34** (1998), no. 2, 155—177

H. Yamane, *On defining relations of affine Lie superalgebras and affine quantized universal enveloping superalgebras*, Publ. RIMS, Kyoto Univ. **35** (1999), 321-–390

Thank you!

Happy Birthday,
Prof. Tarasov and Prof. Varchenko!