Macdonald polynomials and level two Demazure modules for affine $\mathfrak{sl}_{n+1}$

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Notation

(1) $\mathfrak{g}$ is a simple Lie algebra of rank n.
(2) P is the weight lattice and P^+ is the dominant weight lattice of $\mathfrak{g}$.
(3) R is the root lattice and R^+ is the positive root lattice of $\mathfrak{g}$.
(4) $P^+(1) = \{\omega_{i_1} + \cdots + \omega_{i_k} : i_1 < \cdots < i_k \leq n\}$
(5) Given $\lambda \in P^+$, we write $\lambda = 2\lambda_0 + \lambda_1$ where $\lambda_0 \in P^+$ and $\lambda_1 \in P^+(1)$.
(6) For $\lambda \in P^+$, we define $\min(\lambda) = \min\{i \in [1, n] : \lambda(h_i) > 0\}$ and $\max(\lambda) = \max\{i \in [1, n] : \lambda(h_i) > 0\}$
(7) For $\lambda = \sum_{i=1}^{n} a_i\omega_i$, $\mathrm{ht}\lambda = \sum_{i=1}^{n} a_i$ where ω_i's are fundamental weights of $\mathfrak{g}$.
(8) $P_\lambda(z, q, t)$ is the symmetric Macdonald polynomial corresponding to the weight λ.
(9) $s_\lambda(z)$ is the Schur function corresponding to the weight λ.
In this talk, we are only considering the case when $\mathfrak{g} = \mathfrak{sl}_{n+1}$.

Theorem 1 (Biswal-Chari-Shereen-Wand(2019))

There exists a family of polynomials $G_\lambda(z,q) \in \mathbb{C}(q)[z_1,\cdots,z_{n+1}]$ *such that:*

$$G_\lambda(z,q) = \sum_{\mu\leq\lambda} \eta_\lambda^\mu(q) s_\mu(z), \quad \eta_\lambda^\mu(q) \in \mathbb{Z}_+[q], \quad \eta_\lambda^\lambda(q) = 1, \tag{1}$$

$$P_\lambda(z,q,0) = \sum_{\mu\leq\lambda} h_\lambda^\mu(q) G_\mu(z,q), \tag{2}$$

where for $\mu = 2\mu_0 + \mu_1$,

$$h_\lambda^\mu(q) = q^{\frac{1}{2}(\lambda+\mu_1,\ \lambda-\mu)} \prod_{j=1}^{n} \begin{bmatrix} (\lambda-\mu,\ \omega_j) + (\mu_0, \alpha_j) \\ (\lambda-\mu, \omega_j) \end{bmatrix}_q . \tag{3}$$

We prove the above theorem by realizing the polynomial $G_\lambda(z,q)$ as graded character of a finite dimensional module for $\mathfrak{g}[t]$.

Defining polynomials $G_\lambda(z,q)$

For $\lambda = 2\lambda_0 + \lambda_1 \in P^+$, let

$$G_\lambda(z,q) = \sum_{\mu\in P} g_\lambda^\mu(q) P_\mu(z,q,0), \quad g_\lambda^\mu \in \mathbb{Z}[q], \quad \mu \in P^+.$$

where $g_\lambda^\mu(q)$ are uniquely determined by requiring that they satisfy,

$$g_0^\mu = \delta_{\mu,0} \text{ if } \ \mu \in P^+ \ \text{ and } \ g_\lambda^\mu = 0 \ \text{ if } \ \mu \notin P^+,$$

$$g^\mu_{2\lambda_0+2\omega_j} = q^{(2\omega_j, 2\lambda_0+2\omega_j-\mu)} \left(g_{2\lambda_0}^{\mu-2\omega_j} - q^{-(\lambda_0-\mu+\omega_j,\alpha_j)} g_{2\lambda_0}^{\mu-2\omega_j+\alpha_j} \right), j \geq \max \lambda_0, \tag{4}$$

$$g^\mu_{\omega_m+2\lambda_0} = q^{(\omega_m, 2\lambda_0+\omega_m-\mu)} g_{2\lambda_0}^{\mu-\omega_m}, \quad m \in [1,n] \tag{5}$$

and if $\mathrm{ht}\lambda_1 \geq 2$ with $\min \lambda_1 = m$, $\min(\lambda_1 - \omega_m) = p$, then

$$g_\lambda^\mu = q^{(\omega_m, \lambda-\mu)} g_{\lambda-\omega_m}^{\mu-\omega_m} - q^{(\lambda_0, \alpha_{m,p})+1+(\omega_{m-1},\lambda-\mu)} g^{\mu-\omega_{m-1}}_{\lambda-\alpha_{m,p}-\omega_{m-1}}. \tag{6}$$

Definition of $\mathfrak{g}[t]$-modules $W_{\mathrm{loc}}(\lambda)$ and $M(\nu, \lambda)$

- $W_{\mathrm{loc}}(\lambda)$ is the cyclic $\mathfrak{g}[t]$–module generated by an element w_λ with the following relations:

$$(x_i^+ \otimes 1)w_\lambda = 0, \ \ (h \otimes t^r)w_\lambda = \delta_{r,0}\lambda(h)w_\lambda, (x_i^- \otimes 1)^{\lambda(h_i)+1}w_\lambda = 0 \quad (7)$$

for all $i \in [1, n]$ and $h \in \mathfrak{h}$ and $W_{\mathrm{loc}}(\lambda)$ are known to be finite dimensional.

- For $\nu, \lambda \in P^+$ with $\lambda = 2\lambda_0 + \lambda_1$, let $M(\nu, \lambda)$ be the $\mathfrak{g}[t]$–module generated by an element $w_{\nu,\lambda}$ with the following relations:

$$(x_i^+ \otimes 1)w_{\nu,\lambda} = 0, \ \ (h \otimes t^r)w_{\nu,\lambda} = \delta_{r,0}(\lambda + \nu)(h)w_{\nu,\lambda}, \quad (8)$$

$$(x_i^- \otimes 1)^{(\lambda+\nu)(h_i)+1}w_{\nu,\lambda} = 0, \ \ (x_\alpha^- \otimes t^{\nu(h_\alpha)+\lceil \lambda(h_\alpha)/2 \rceil})w_{\nu,\lambda} = 0, \quad (9)$$

for all $i \in [1, n]$, $h \in \mathfrak{h}$ and $\alpha \in R^+$.

Graded character

Both $W_{\mathrm{loc}}(\lambda)$ and $M(\nu,\lambda)$ belong to the category of finite–dimensional $\mathbb{Z}_+$–graded modules for $\mathfrak{g}[t]$. An object of this category is a finite-dimensional module V for $\mathfrak{g}[t]$ which admits a compatible $\mathbb{Z}$–grading i.e.,

$$V=\bigoplus_{s\in\mathbb{Z}} V[s], \quad (x\otimes t^r)V[s]\subset V[r+s], \quad x\in\mathfrak{g}, \quad r\in\mathbb{Z}_+.$$

For any $p\in\mathbb{Z}$ we let $\tau_p^* V$ be the graded module which is given by shifting the grades up by p and leaving the action of $\mathfrak{g}[t]$ unchanged. The morphisms between graded modules are $\mathfrak{g}[t]$-maps of grade zero. Clearly for any object V of this category the subspace $V[s]$ is a $\mathfrak{g}$–module and the graded character of V is the element of $\mathbb{Z}[q,q^{-1}][P]$ given by:

$$\mathrm{ch}_{\mathrm{gr}} V=\sum_{s\in\mathbb{Z}} q^s \mathrm{ch}\, V[s]=\sum_{\mu\in P^+}\sum_{s\in\mathbb{Z}} \dim\mathrm{Hom}_{\mathfrak{g}}(V(\mu),V[s])q^s \mathrm{ch}\, V(\mu).$$

Demazure modules

Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \bigoplus \mathbb{C}c \bigoplus \mathbb{C}d$ be an affine Lie algebra of rank $n+1$ and $V(\Lambda)$ be an irreducible integrable representation of $\hat{\mathfrak{g}}$. Then for an affine Weyl group element w, the extremal weight space $V(\Lambda)_{w\Lambda}$ of $V(\Lambda)$ is one dimensional. Let $v_{w\Lambda} \in V(\Lambda)$. Then the Demazure module $D_\omega(\Lambda) = \mathbb{U}(\mathfrak{b})v_{w\Lambda}$ where $\mathfrak{b}$ is the Borel subalgebra of $\hat{\mathfrak{g}}$ and $\mathbb{U}(\mathfrak{b})$ is the universal enveloping algebra of $\mathfrak{b}$. But $D_w(\Lambda)$ is not stable under the action of $\mathfrak{g}[t]$. $D_w(\Lambda)$ is $\mathfrak{g}[t]$-stable iff $w\Lambda(h_i) \leq 0$ for $1 \leq i \leq n$. Hence $w\Lambda = \ell\Lambda_0 + \omega_0\lambda + m\delta$ for some $\ell \in \mathbb{Z}_+, \lambda \in P^+, m \in \mathbb{Z}$ where w_0 is the longest Weyl group element of $\mathfrak{g}$. We denote such a Demazure module by $\tau_m^* D(\ell, \lambda)$. If $m = 0$, we simply denote it by $D(\ell, \lambda)$. The modules $D(\ell, \lambda)$ are always finite-dimensional.

Motivation for $\mathfrak{g}[t]$-stable Demazure modules

- The characters of level one Demazure modules $D(1, \lambda)$ is equal to the specialization of symmetric Macdonald polynomials $P_\lambda(X, q, t)$ at $t = 0$.
- $D(1, \lambda)$ is isomorphic to standard modules of Nakajima Quiver varieties.
- $D(1, \lambda)$ and $D(2, \lambda)$ appear as graded limits of tensor product of special classes of irreducible representations of quantum affine algebras.

The module $M(\nu,\lambda)$ is a $\mathbb{Z}_+$-graded $\mathfrak{g}[t]$–module once we declare the grade of $w_{\nu,\lambda}$ to be zero. In the case when $\lambda = 0$, it is clear that the relation in $M_{\nu,0}$ is a consequence of the relations in local Weyl module; in particular the module $M(\nu, 0)$ is just the local Weyl module, which is denoted as $W_{\text{loc}}(\nu)$. The local Weyl modules are known to be finite–dimensional. Since $M(\nu,\lambda)$ is obviously a quotient of $W_{\text{loc}}(\nu+\lambda)$ it follows that $M(\nu,\lambda)$ is also finite–dimensional. Moreover

$$\dim \text{Hom}_{\mathfrak{g}}(V(\mu), M(\nu,\lambda)) \neq 0$$

$$\implies \quad \nu+\lambda-\mu \in Q^+, \;\; \dim \text{Hom}_{\mathfrak{g}}(V(\nu+\lambda), M(\nu,\lambda)) = 1.$$

It is clear that the elements of the set $\{\mathrm{ch}_{\mathrm{gr}} M(\mu, 0) : \mu \in P^+\}$ (resp. of the set $\{\mathrm{ch}_{\mathrm{gr}} M(0, \mu) : \mu \in P^+\}$) are linearly independent and that their $\mathbb{Z}[q, q^{-1}]$ span contains $\mathrm{ch}\, V(\lambda)$, $\lambda \in P^+$. Hence we can write

$$\mathrm{ch}_{\mathrm{gr}} M(\nu, \lambda) = \sum_{\mu \in P^+} g^{\mu}_{\nu,\lambda}(q) \mathrm{ch}_{\mathrm{gr}} M(\mu, 0) = \sum_{\mu \in P^+} h^{\mu}_{\nu,\lambda}(q) \mathrm{ch}_{\mathrm{gr}} M(0, \mu),$$

where

$$g^{\nu+\lambda}_{\nu,\lambda} = 1 = h^{\nu+\lambda}_{\nu,\lambda}, \quad g^{\mu}_{\nu,\lambda} = h^{\mu}_{\nu,\lambda} = 0 \text{ if } \lambda + \nu - \mu \notin Q^+.$$

Moreover the linear independence also implies that for all $\nu, \mu \in P^+$,

$$\sum_{\mu' \in P^+} h^{\mu'}_{\nu,0} g^{\mu}_{0,\mu'} = \delta_{\nu,\mu} = \sum_{\mu' \in P^+} g^{\mu'}_{0,\nu} h^{\mu}_{\mu',0}. \tag{10}$$

It is known that $W_{\mathrm{loc}}(\nu)$ (equivalently $M(\nu, 0)$) is graded isomorphic to a Demazure module occurring in a level one representation of the affine Lie algebra $\mathfrak{sl}_{n+1}$. In particular using a result of Sanderson and Ion, it follows that

$$\mathrm{ch}_{\mathrm{gr}} M(\nu, 0) = P_{\nu}(z, q, 0). \tag{11}$$

admissible pair of dominant weights

We say that a pair $(\nu, \lambda) \in P^+ \times P^+$ is admissible if one of the following hold: write $\lambda = 2\lambda_0 + \lambda_1$, $\nu = 2\nu_0 + \nu_1$; then either

- $\lambda_1 = 0$, or
- $\lambda_1 \neq 0$, $\nu_0 = \omega_i$ for some $i \in [0, n]$ with $\max \nu_1 < \min \lambda_1$ and if $i \in [1, n]$ we also require that $i < \min \lambda_1 - 1$ and $\nu_1(h_i) = \nu_1(h_{i+1}) = 0$.

Key tool

The proof of Theorem 1 is using representation theory. The main tool is the following three short exact sequences. Let (ν,λ) be admissible.

- If $j \in [1,n]$ is such that $\nu(h_j) \geq 2$, then

  $$0 \to \tau^*_{(\lambda_0+\nu)(h_j)-1} M(\nu-\alpha_j,\lambda) \to M(\nu,\lambda) \to M(\nu-2\omega_j,\lambda+2\omega_j) \to 0.$$

- If $\nu_0 = 0$ and $\max\nu_1 = m$ and $\min\lambda_1 = p > 0$ then
  $0 \to \tau^*_{\lambda_0(h_{m,p})+1} M(\nu-\omega_m+\omega_{m-1},\lambda-\omega_p+\omega_{p+1}) \to M(\nu,\lambda) \to M(\nu-\omega_m,\lambda+\omega_m) \to 0.$
- If $\lambda \in P^+(1)$ and $m \in [0,n]$ with $m < \min\lambda$ for $\lambda \neq 0$, then

  $$0 \to \tau^*_1 M(\omega_{m-1},\lambda+\omega_{m+1}) \to M(\omega_m,\lambda+\omega_m) \to D(2,\lambda+2\omega_m) \to 0.$$

and we also use the following fact: Given (ν,λ) admissible and $\mu \in P^+$ we have

$$g^{\mu}_{\nu,\lambda} = q^{(\lambda+\nu-\mu,\nu)} g^{\mu-\nu}_{\lambda}.$$

Theorem 2 (Biswal, Chari, Shereen, Wand(2019))

For admissible pairs (ν, λ), *the following holds:*

$$M(\nu, \lambda) \cong D(1, \nu) * D(2, \lambda)$$

In particular,

$$M(0, \lambda) \cong D(2, \lambda), M(\nu, 0) \cong D(1, \nu)$$

The following corollary tells us that $G_\lambda(z, q)$ are characters of level two Demazure modules $D(2, \lambda)$.

Corollary 3

For $\lambda, \nu \in P^+$ *we have*

$$ch_{gr} M(0, \lambda) = G_\lambda(z, q), \quad i.e. \;\; g^{\mu}_{0,\lambda}(q) = g^{\mu}_{\lambda}(q).$$

$$ch_{gr} M(\nu, 0) = P_\nu(z, q, 0)$$

Theorem 4 (Katsuyuki Naoi(2010))

Let $\mathfrak{g}$ be a simple Lie algebra. If $m \geq \ell$, then $D(\ell, \lambda)$ admits a filtration by level m-Demazure modules i.e there exists a sequece

$$(0) \subseteq V_0 \subseteq V_1 \subseteq \cdots \subseteq V_r = D(\ell, \lambda)$$

of graded submodules such that each successive quotient $\frac{V_i}{V_{i-1}}$ is isomorphic to some Demazure module of level m..

Notation

- Numerical Multiplicity:
  $[D(\ell,\lambda) : D(m,\mu)]$=The number of successive quotients that are isomorphic to the module $D(m,\mu)$.
- Graded or q-multiplicity(reduces to numerical multiplicity at $q=1$):
  $[D(\ell,\lambda) : D(m,\mu)]_q = \sum_{i:\frac{V_i}{V_{i-1}}\cong D(m,\mu)} q^{min\ grade\ V_i}$
- Independent of the filtration.
- $[D(\ell,\lambda) : D(m,\mu)] \neq 0$ implies $\lambda-\mu \in R^+$.

As a consequence of our main theorem, we get the following corollary:

Corollary 5

For $\mu = 2\mu_0 + \mu_1$,

$$[D(1,\lambda) : D(2,\mu)]_q = q^{\frac{1}{2}(\lambda+\mu_1,\ \lambda-\mu)} \prod_{j=1}^{n} \begin{bmatrix} (\lambda-\mu,\ \omega_j) + (\mu_0, \alpha_j) \\ (\lambda-\mu, \omega_j) \end{bmatrix}_q$$

connections to number theory

(1) $\sum_{k=0}^{\infty}[D(1,(m+2k)\omega) : D(3,m\omega)]_q x^k$ are known to be mock theta functions after specializing x to integer powers of q in the case $\mathfrak{g}=\mathfrak{sl}_2$.
(2) $\sum_{\alpha\in R^+}[D(1,\alpha) : D(2,0)]X^{\alpha}$ are also cone theta functions.
(3) Is there any connection of $\sum_{\mu\in P^+}[D(1,\lambda) : D(m,\mu)]X^{\lambda-\mu}$ to mock modular forms for any $m>1$?

Further questions

- What is the combinatorial interpretation of the polynomials $G_\lambda(z, q)$ and $\eta_\lambda^\mu(q)$?
- Is there any geometric interpretation of the coefficients of powers of q in the polynomials $[D(\ell, \lambda) : D(m, \mu)]_q$ for $m \geq \ell$?
- Is polynomial coming from the character of $\mathfrak{g}[t]$-modules $M(\lambda, \mu)$ related to some well-known polynomials now that we know them for the extreme cases either for $\lambda = 0$ or for $\mu = 0$?

Reference:

- Rekha Biswal, Vyjayanthi Chari, Lisa Schneider, Sankaran Viswanath; *"Demazure flags, Chebyshev polynomials, Partial and Mock Theta functions"*, arxiv: 1502.05322, (journal of Combinatorial Theory Series A).
- Rekha Biswal, Vyjayanthi Chari and Deniz Kus; *"Demazure flags, q–Fibonacci polynomials and hypergeometric series"* (Res. Math. Sci)
- Rekha Biswal, Vyjayanthi Chari, Peri Shereen and Jeffrey Wand; *"Macdonald polynomials and level two Demazure modules for affine $\mathfrak{sl}_{n+1}$"* (In preparation)
- Rekha Biswal and Deniz Kus; *"A combinatorial formulae for graded multiplicities in excellent filtrations"*, (to appear in Transformation groups)

Thank you