Quasi-particle bases of principal subspaces of representations of affine Lie algebras

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- This talk is based on a joint work with Slaven Kožić

arXiv:1902.10794

Znanstveni centar izvrsnosti
za kvantne i kompleksne sustave te
reprezentacije Liejevih algebri

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Affine Lie algebra of type $F_4^{(1)}$

- $\mathfrak{g}$ - simple Lie algebra of type F_4

$$\underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} \Rightarrow \underset{\alpha_3}{\circ} - \underset{\alpha_4}{\circ}$$

  - $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$
  - $\mathfrak{n}_+ = \oplus_{\alpha \in R_+} \mathfrak{n}_\alpha, \ \mathfrak{n}_\alpha = \mathbb{C} x_\alpha, \ \ \alpha \in R_+$

- $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ affine Kac-Moody Lie algebra of type $F_4^{(1)}$
  - $[x(j_1), y(j_2)] = [x, y]\,(j_1 + j_2) + \langle x, y \rangle\, j_1 \delta_{j_1+j_2,0}\, c, \quad [c, x(j)] = 0,$ $[d, x(j)] = jx(j)$, where $x(j) = x \otimes t^j$ for $x \in \mathfrak{g}$ and $j \in \mathbb{Z}$
  - $\widetilde{\mathfrak{n}}_+ = \mathfrak{n}_+ \otimes \mathbb{C}[t, t^{-1}]$
  - Λ_i, $i = 0, 1, \ldots, 4$ - fundamental weights

Modules of affine Lie algebra

$k \in \mathbb{N}$

$N(k\Lambda_0)$ - generalized Verma module

$$N(k\Lambda_0) = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_+)} \mathbb{C}v_{N(k\Lambda_0)} \overset{\text{PBW}}{\cong} U(\tilde{\mathfrak{g}}_-)$$

- $\tilde{\mathfrak{g}}_+ = \bigoplus_{n\geq 0}(\mathfrak{g} \otimes t^n) \oplus \mathbb{C}c \oplus \mathbb{C}d$, $\tilde{\mathfrak{g}}_- = \bigoplus_{n<0}(\mathfrak{g} \otimes t^n)$ - subalgebras of $\tilde{\mathfrak{g}}$
- $1 \otimes v_{N(k\Lambda_0)} = v_N$ - highest weight vector
- a vertex operator algebra with a vacuum vector v_N, with a vertex operator map

$$\begin{aligned} Y(\cdot, z) : N(k\Lambda_0) &\rightarrow \text{End}\, N(k\Lambda_0)\left[\left[z, z^{-1}\right]\right] \\ x &\mapsto Y(x(-1)v_N, z) = \sum_{m\in\mathbb{Z}} x_m z^{-m-1} = x(z), \ x \in \mathfrak{g} \end{aligned}$$

satisfying certain properties and with a conformal vector.

$L(k\Lambda_0)$ - standard (integrable highest weight) $\tilde{\mathfrak{g}}$-module

- v_L - a highest weight vector of $L(k\Lambda_0)$
- simple vertex operator algebra
- every level k standard $\tilde{\mathfrak{g}}$-module is $L(k\Lambda_0)$-module

Principal subspace

- $V = N(k\Lambda_0)$ or $V = L(k\Lambda_0)$

Principal subspace of V

$$W_V = U(\widetilde{\mathfrak{n}}_+)v$$

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Principal subspace of V

$$W_V = U(\widetilde{\mathfrak{n}}_+)v$$

Character of the principal subspace

$$ch\ W_V = \sum_{m,r_1,\ldots,r_4 \geq 0} dim\,(W_V)_{-m\delta+r_1\alpha_1+\ldots+r_4\alpha_4} q^m \prod_{i=1}^{4} y_i^{r_i}$$

- $(W_V)_{-m\delta+r_1\alpha_1+\ldots+r_4\alpha_4}$ the weight subspaces of W_V with respect to $\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$

Motivation

- principal subspaces were first introduced and studied by Feigin-Stoyanovsky

  B. Feigin and A. Stoyanovsky, Quasi-particles models for the representations of Lie algebras and geometry of flag manifold; arXiv:hep-th/9308079.

- combinatorial bases in $x_{\alpha_i}(m)$ - quasi-particle of color i, charge 1 and energy $-m$

- energies of basis monomial $x_{\alpha_i}(m_2)x_{\alpha_i}(m_1)$ satisfy difference two condition if $m_2 \leq m_1 - 2$

- if $\tilde{\mathfrak{g}}$ is of type $A_1^{(1)}$ we have a connection of ch $W_{L(\Lambda_i)}$, $i = 0, 1$ with Rogers-Ramanujan identities

  - ch $W_{L(\Lambda_0)} = \sum_{r\geq 0} \frac{q^{r^2}}{(q;q)_r} = \prod_{i\geq 0} \frac{1}{(1-q^{5i+1})(1-q^{5i+4})}$, where $(q;q)_r = \prod_{i=1}^{r}(1-q^i)$

  - ch $W_{L(\Lambda_1)} = \sum_{r\geq 0} \frac{q^{r^2+r}}{(q;q)_r} = \prod_{i\geq 0} \frac{1}{(1-q^{5i+2})(1-q^{5i+3})}$

Quasi-particles

- Georgiev - constructed combinatorial bases of principal subspaces of some standard modules of $A_l^{(1)}$ in terms of quasi-particles of higher charges

G. Georgiev, *Combinatorial constructions of modules for infinite-dimensional Lie algebras, I. Principal subspace*, J. Pure Appl. Algebra **112** (1996), 247–286; arXiv:hep-th/9412054.

- $r \in \mathbb{N}$, $m \in \mathbb{Z}$

Quasi-particle of color *i*, charge *r* and energy -*m*

$$x_{r\alpha_i}(m) = \sum_{\substack{m_1,\ldots,m_r \in \mathbb{Z} \\ m_1+\cdots+m_r=m}} x_{\alpha_i}(m_r)\cdots x_{\alpha_i}(m_1)$$

$$x_{r\alpha_i}(z) = Y(x_{\alpha_i}(-1)^r v_N, x) = \sum_{m\in\mathbb{Z}} x_{r\alpha_i}(m) z^{-m-r}$$

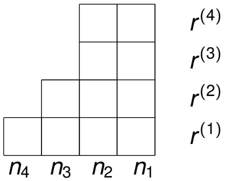
Quasi-particle monomials

- $b(\alpha_4)b(\alpha_3)b(\alpha_2)b(\alpha_1)v$ elements of bases of W_V where

$$b(\alpha_i) = x_{n_{r^{(1)},i}\alpha_i}(m_{r^{(1)},i}) \cdots x_{n_{2,i}\alpha_i}(m_{2,i})x_{n_{1,i}\alpha_1}(m_{1,i})$$

- charge-type $\left(n_{r^{(1)},i}, \ldots, n_{1,i}\right)$; $\quad 0 \leq n_{r^{(1)},i} \leq \cdots \leq n_{1,i}$,
  - color-type r_i; $\qquad \sum_{p=1}^{r_i^{(1)}} n_{p,i} = r_i$
- dual-charge-type $(r_i^{(1)}, r_i^{(2)}, \ldots, r_i^{(s)})$;

$$r_i^{(1)} \geq r_i^{(2)} \geq \ldots \geq r_i^{(s)} \geq 0, \quad \sum_{p=1}^{s} r_i^{(p)} = r_i$$



Character of $W_{L(k\Lambda_0)}$

- relations among quasi-particles: $x_{n_i\alpha_i}(m_i)x_{n_j\alpha_j}(m_j)$, where $1 \leq i, j \leq 4$
- energies of quasi-particle monomials from basis satisfy difference conditions, which we write in terms of dual-charge type

Theorem (B., S. Kožić)

$$\mathrm{ch}\, W_{L(k\Lambda_0)} = \sum_{\mathcal{R}'} \prod_{i=1}^{4} F_{\mathcal{R}'_i}(q) I_{\mathcal{R}'_i,\mathcal{R}'_{i+1}}(q) \prod_{i=1}^{4} y_i^{r_i},$$

where

$$\mathcal{R}' = (\mathcal{R}'_4, \mathcal{R}'_3, \mathcal{R}'_2, \mathcal{R}'_1),$$

$$\mathcal{R}'_i = (r_i^{(1)}, r_i^{(2)}, \ldots, r_i^{(k)}), \quad \textit{for} \;\; i = 1, 2,$$

$$\mathcal{R}'_i = (r_i^{(1)}, r_i^{(2)}, \ldots, r_i^{(2k)}), \quad \textit{for} \;\; i = 3, 4,$$

$$F_{\mathcal{R}'_i}(q) = \frac{q^{\sum_{t=1}^{k} r_i^{(t)^2}}}{(q;q)_{r_i^{(1)}-r_i^{(2)}} \cdots (q;q)_{r_i^{(k)}}}, \quad \textit{for} \;\; i = 1, 2$$

$$F_{\mathcal{R}'_i}(q) = \frac{q^{\sum_{t=1}^{2k} r_i^{(t)^2}}}{(q;q)_{r_i^{(1)}-r_i^{(2)}} \cdots (q;q)_{r_i^{(2k)}}}, \quad \textit{for} \;\; i = 3, 4$$

$$I_{\mathcal{R}'_1,\mathcal{R}'_2}(q) = q^{-\sum_{t=1}^{k} r_1^{(t)} r_2^{(t)}},$$

$$I_{\mathcal{R}'_2,\mathcal{R}'_3}(q) = q^{-\sum_{t=1}^{k} r_2^{(t)}(r_3^{(2t-1)} + r_3^{(2t)})},$$

$$I_{\mathcal{R}'_3,\mathcal{R}'_4}(q) = q^{-\sum_{t=1}^{2k} r_3^{(t)} r_4^{(t)}}.$$

Character of $W_{N(k\Lambda_0)}$

Theorem (B., S. Kožić)

$$\mathrm{ch}\ W_{N(k\Lambda_0)} = \sum_{\substack{\mathcal{R}'_{u_i} \\ u_1,u_2,u_3,u_4\geqslant 0}} \prod_{i=1}^{4} F_{\mathcal{R}'_i}(q) I_{\mathcal{R}'_i,\mathcal{R}'_{i+1}}(q) \prod_{i=1}^{4} y_i^{r_i},$$

where

$$\mathcal{R}'_{u_i} = (\mathcal{R}'_4, \mathcal{R}'_3, \mathcal{R}'_2, \mathcal{R}'_1),$$

$$\mathcal{R}'_i = (r_i^{(1)}, r_i^{(2)}, \ldots, r_i^{(u_i)}), \quad \textit{for}\ \ i = 1, 2,$$

$$\mathcal{R}'_i = (r_i^{(1)}, r_i^{(2)}, \ldots, r_i^{(2u_i)}), \quad \textit{for}\ \ i = 3, 4,$$

$$F_{\mathcal{R}'_i}(q) = \frac{q^{\sum_{t=1}^{u_i} r_i^{(t)2}}}{(q;q)_{r_i^{(1)}-r_i^{(2)}} \cdots}, \quad \textit{for}\ \ i = 1, 2$$

$$F_{\mathcal{R}'_i}(q) = \frac{q^{\sum_{t=1}^{2u_i} r_i^{(t)2}}}{(q;q)_{r_i^{(1)}-r_i^{(2)}} \cdots}, \quad \textit{for}\ \ i = 3, 4$$

$$I_{\mathcal{R}'_1,\mathcal{R}'_2}(q) = q^{-\sum_{t=1}^{k} r_1^{(t)} r_2^{(t)}},$$

$$I_{\mathcal{R}'_2,\mathcal{R}'_3}(q) = q^{-\sum_{t=1}^{k} r_2^{(t)}(r_3^{(2t-1)} + r_3^{(2t)})},$$

$$I_{\mathcal{R}'_3,\mathcal{R}'_4}(q) = q^{-\sum_{t=1}^{2k} r_3^{(t)} r_4^{(t)}}.$$

Character of $W_{N(k\Lambda_0)}$

- $\widetilde{\mathfrak{n}}_+^{<0} = \mathfrak{n}_+ \otimes t^{-1}\mathbb{C}\left[t^{-1}\right]$

Isomorphism of $\widetilde{\mathfrak{n}}_+^{<0}$-modules

$$W_{N(k\Lambda_0)} \cong U(\widetilde{\mathfrak{n}}_+^{<0})$$

Theorem (B., S. Kožić)

$$\frac{1}{\prod_{\alpha\in R_+}(\alpha;q)_\infty} =$$

$$= \sum_{\substack{r_1^{(1)}\geqslant r_1^{(2)}\geqslant\cdots\geqslant 0\\ r_2^{(1)}\geqslant r_2^{(2)}\geqslant\cdots\geqslant 0}} \frac{q^{\sum_{i=1}^{2}\sum_{t\geq 1} r_i^{(t)^2} - \sum_{t\geq 1} r_1^{(t)} r_2^{(t)}}}{\prod_{i=1}^{2}(q)_{r_i^{(1)}-r_i^{(2)}}\cdots} \prod_{i=1}^{2} y_i^{n_i}$$

$$\times \sum_{\substack{r_3^{(1)}\geqslant r_3^{(2)}\geqslant\cdots\geqslant 0\\ r_4^{(1)}\geqslant r_4^{(2)}\geqslant\cdots\geqslant 0}} \frac{q^{\sum_{i=3}^{4}\sum_{t\geq 1} r_i^{(t)^2} - \sum_{t\geq 1} r_3^{(t)} r_4^{(t)} - \sum_{t\geq 1} r_2^{(t)}(r_3^{(2t-1)}+r_3^{(2t)})}}{\prod_{i=3}^{4}(q)_{r_i^{(1)}-r_i^{(2)}}\cdots} \prod_{i=3}^{4} y_i^{n_i},$$

where $(\alpha;q)_\infty = (qy_1^{a_1};q)_\infty\cdots(qy_4^{a_4};q)_\infty$, *for* $\alpha = \sum_{i=1}^{4} a_i\alpha_i$.

- generalization of the theorem of Euler and Cauchy:

$$\frac{1}{(yq)_\infty} = \sum_{m=0}^{\infty} \frac{q^{m^2}y^m}{(q)_m(yq)_m}$$

Thank you!