Reduction of a bi-Hamiltonian hierarchy on $T^*\mathrm{U}(n)$ to spin Ruijsenaars–Sutherland models

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Consider the following hierarchy of evolution equations:

$$\dot{Q}_j = (\mathrm{i}L^k Q)_{jj}, \quad \dot{L} = [\mathcal{R}(Q)(\mathrm{i}L^k), L], \qquad \text{for} \quad (Q, L) \in \mathbb{T}^n_{\mathrm{reg}} \times \mathrm{i}\mathfrak{u}(n), \ \forall k \in \mathbb{N}.$$

L is an $n \times n$ Hermitian matrix, $Q \equiv \mathrm{diag}(Q_1, Q_2, \ldots, Q_n)$ is a diagonal unitary matrix, and $\mathcal{R}(Q)$ is the dynamical r-matrix given below.

There is a gauge freedom in this system:

$$(Q, L) \Longleftrightarrow (\eta Q \eta^{-1}, \eta L \eta^{-1}) \qquad \forall \eta \in \mathcal{N}(n) := N_{\mathbb{T}^n}(\mathrm{U}(n)).$$

The evolutional derivations of gauge invariant 'observables' commute due to the CDYBE satisfied by the dynamical r-matrix: $\mathcal{R}(Q) := 0$ on the Cartan subalgebra $\mathfrak{gl}(n, \mathbb{C})_0 < \mathfrak{gl}(n, \mathbb{C})$ and

$$\mathcal{R}(Q) := \frac{1}{2}(\mathrm{Ad}_Q + \mathrm{id})(\mathrm{Ad}_Q - \mathrm{id})^{-1} \text{ on } \mathfrak{gl}(n, \mathbb{C})_\perp, \quad (\mathrm{Ad}_Q(X) := QXQ^{-1}).$$

Plan: First, I exhibit a bi-Hamiltonian structure for this system. Then, if time permits, I shall explain why I call it 'spin Ruijsenaars–Sutherland hierarchy'. For details, see `arXiv:1908.02467 [math-ph]`

Recall celebrated exactly solvable many-body models

Trigonometric Sutherland system:

$$H_{\text{Suth}} = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k} \frac{x^2}{\sin^2(q_k - q_j)}$$

Trigonometric Ruijsenaars–Schneider system:

$$H_{\text{RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j\neq k} \left[1 + \frac{x^2}{\sin^2(q_k - q_j)}\right]^{\frac{1}{2}}$$

Light-cone Hamiltonians of the RS system:

$$H_{\pm} = \sum_{k=1}^{n} e^{\pm p_k} \prod_{j\neq k} \left[1 + \frac{x^2}{\sin^2(q_k - q_j)}\right]^{\frac{1}{2}}$$

Describe integrable interactions of n points moving on the circle.

Generalize rational Calogero–Moser model of points on the real line:

$$H_{\text{CM}} = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k} \frac{x^2}{(q_k - q_j)^2}$$

Bi-Hamiltonian hierarchy on $T^*\mathrm{U}(n)$: We start with the manifold

$$\mathfrak{M} := \mathrm{U}(n) \times \mathrm{i}\mathfrak{u}(n) := \{(g, L) \mid g \in \mathrm{U}(n), L \in \mathrm{i}\mathfrak{u}(n)\}.$$

We use the real Lie algebra $\mathfrak{gl}(n, \mathbb{C})$, equipped with the bilinear form

$$\langle X, Y\rangle := \Im \mathrm{tr}(XY), \quad \forall X, Y \in \mathfrak{gl}(n, \mathbb{C}),$$

and the real vector space decomposition (Manin triple)

$$\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) + \mathfrak{b}(n)$$

$$\text{with} \quad \mathfrak{b}(n) := \mathrm{span}_{\mathbb{R}}\{E_{jj}, E_{kl}, \mathrm{i}E_{kl} \mid 1 \leq j \leq n,\ 1 \leq k < l \leq n\}.$$

This gives the decomposition $X = X_{\mathfrak{u}(n)} + X_{\mathfrak{b}(n)}$ for every $X \in \mathfrak{gl}(n, \mathbb{C})$.
For a real function $F \in C^\infty(\mathfrak{M})$, the derivatives

$$D_1F, D_1'F \in C^\infty(\mathfrak{M}, \mathfrak{b}(n)),\ d_2F \in C^\infty(\mathfrak{M}, \mathfrak{u}(n)) \quad \text{are defined by}$$

$$\frac{d}{dt}\Big|_{t=0} F(e^{tX} g e^{tX'}, L + tY) = \langle D_1F(g, L), X\rangle + \langle D_1'F(g, L), X'\rangle + \langle d_2F(g, L), Y\rangle$$

for all $X, X' \in \mathfrak{u}(n)$ and $Y \in \mathrm{i}\mathfrak{u}(n)$.

Proposition 1. *The following formulas define two Poisson brackets on* $C^\infty(\mathfrak{M}, \mathbb{R})$*:*

$$\{F, H\}_1(g, L) = \langle D_1 F, d_2 H\rangle - \langle D_1 H, d_2 F\rangle + \langle L, [d_2 F, d_2 H]\rangle\,,$$

and

$$\begin{aligned}\{F, H\}_2(g, L) &= \langle D_1 F, L d_2 H\rangle - \langle D_1 H, L d_2 F\rangle \\ &\quad + 2\left\langle L d_2 F, (L d_2 H)_{\mathfrak{u}(n)}\right\rangle - \frac{1}{2}\left\langle D_1' F, g^{-1}(D_1 H) g\right\rangle,\end{aligned}$$

where the derivatives are taken at the point (g, L)*.*

Remark: The first bracket is the canonical Poisson bracket of the cotangent bundle, expressed in terms of right-trivialization and taking $\mathrm{i}\mathfrak{u}(n)$ and $\mathfrak{b}(n)$ as models of $\mathfrak{u}(n)^*$. The restriction of the second bracket to the open submanifold $\mathrm{U}(n) \times \exp(\mathrm{i}\mathfrak{u}(n)) \subset \mathfrak{M}$ is a convenient multiple of Semenov-Tian-Shansky's non-degenerate Poisson bracket on the Heisenberg double of the standard Poisson–Lie group $\mathrm{U}(n)$.

$\Big[$Remark: $\mathrm{GL}(n, \mathbb{C}) \ni K = b_L g_R^{-1} = g_L b_R^{-1} \mapsto (g_R, b_R b_R^\dagger) \in \mathrm{U}(n) \times \exp(\mathrm{i}\mathfrak{u}(n))\Big]$

Introduce the vector field $\mathcal{D}$ on $\mathfrak{M}$ that acts as the following derivation of the 'coordinate functions'

$$\mathcal{D}[g_{ij}] := 0, \quad \mathcal{D}[L_{ij}] := \delta_{ij}.$$

Its flow through $(g(0), L(0))$ reads $(g(t), L(t)) = (g(0), L(0) + t\mathbf{1}_n)$.

Proposition 2. *For $F \in C^\infty(\mathfrak{M})$, let $\mathcal{D}[F]$ denote the derivative along the vector field $\mathcal{D}$. The Poisson brackets on $C^\infty(\mathfrak{M})$ satisfy*

$$\{F, H\}_1 = \{F, H\}_2^{\mathcal{D}} \equiv \mathcal{D}[\{F, H\}_2] - \{\mathcal{D}[F], H\}_2 - \{F, \mathcal{D}[H]\}_2,$$

$$\{F, H\}_1^{\mathcal{D}} \equiv \mathcal{D}[\{F, H\}_1] - \{\mathcal{D}[F], H\}_1 - \{F, \mathcal{D}[H]\}_1 = 0,$$

and thus they define an exact bi-Hamiltonian structure.

The Hamiltonians $H_k(g, L) := \frac{1}{k}\mathrm{tr}(L^k)$ $(\forall k \in \mathbb{N})$ satisfy

$$\{F, H_k\}_2 = \{F, H_{k+1}\}_1$$

and induce the bi-Hamiltonian 'free flows'

$$(g(t), L(t)) = \left(\exp(\mathrm{i}tL(0)^k)g(0), L(0)\right).$$

Consider the following action of the group $\mathsf{U}(n)$ on $\mathfrak{M}$:

$$A_\eta(g, L) = (\eta g \eta^{-1}, \eta L \eta^{-1}), \qquad \forall \eta \in \mathsf{U}(n),\ (g, L) \in \mathfrak{M}.$$

One can show that the ring of invariant functions is closed under both Poisson brackets.

Lemma 3. *The Poisson brackets* $\{\ ,\ \}_1$ *and* $\{\ ,\ \}_2$ *on* $C^\infty(\mathfrak{M})$ *induce two compatible Poisson brackets on* $C^\infty(\mathfrak{M})^{\mathsf{U}(n)}$.

Noting that H_k is $\mathsf{U}(n)$ invariant, we can perform Poisson reduction,i.e., take quotient by $\mathsf{U}(n)$. From now on we restrict our attention to the dense open subset

$$\mathfrak{M}_{\mathsf{reg}} := \mathsf{U}(n)_{\mathsf{reg}} \times \mathrm{i}\mathfrak{u}(n).$$

Every $\mathsf{U}(n)$ orbit in $\mathfrak{M}_{\mathsf{reg}}$ contains representatives in the submanifold

$$S := \mathbb{T}^n_{\mathsf{reg}} \times \mathrm{i}\mathfrak{u}(n) \subset \mathfrak{M}_{\mathsf{reg}} \qquad \text{('gauge slice')}$$

and this submanifold is preserved by the action of the normalizer, $\mathcal{N}(n)$, of $\mathbb{T}^n$ in $\mathsf{U}(n)$. The embedding $\iota : \mathbb{T}^n_{\mathsf{reg}} \times \mathrm{i}\mathfrak{u}(n) \to \mathfrak{M}_{\mathsf{reg}}$ yields the identification

$$C^\infty(\mathfrak{M}_{\mathsf{reg}})^{\mathsf{U}(n)} \simeq C^\infty(\mathbb{T}^n_{\mathsf{reg}} \times \mathrm{i}\mathfrak{u}(n))^{\mathcal{N}(n)} \qquad \text{('restricted invariants')}$$

We obtain the reduced Poisson algebras $\left(C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathrm{i}\mathfrak{u}(n))^{\mathcal{N}(n)}, \{\ ,\ \}_i^{\mathrm{red}}\right)$:

$$\{F \circ \iota, H \circ \iota\}_i^{\mathrm{red}} := \{F, H\}_i \circ \iota \quad \text{for} \quad F, H \in C^\infty(\mathfrak{M}_{\mathrm{reg}})^{\mathrm{U}(n)},\ i = 1, 2.$$

Using $\mathcal{R}(Q) \in \mathrm{End}(\mathfrak{gl}(n, \mathbb{C}))$, introduce

$$[X, Y]_{\mathcal{R}(Q)} := [\mathcal{R}(Q)X, Y] + [X, \mathcal{R}(Q)Y], \quad \forall X, Y \in \mathfrak{gl}(n, \mathbb{C}).$$

For any $f \in C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathrm{i}u(n))$, we have the $\mathfrak{b}(n)_0$-valued derivative $D_1 f$ and the $\mathfrak{u}(n)$-valued derivative $d_2 f$:

$$\langle D_1 f(Q, L), X\rangle + \langle d_2 f(Q, L), Y\rangle = \frac{d}{dt}\Big|_{t=0} f(e^{tX}Q, L + tY).$$

Theorem 4. *For $f, h \in C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathrm{i}\mathfrak{u}(n))^{\mathcal{N}(n)}$, the reduced Poisson brackets obey the explicit formulas*

$$\{f, h\}_1^{\mathrm{red}}(Q, L) = \langle D_1 f, d_2 h\rangle - \langle D_1 h, d_2 f\rangle + \langle L, [d_2 f, d_2 h]_{\mathcal{R}(Q)}\rangle,$$

and

$$\{f, h\}_2^{\mathrm{red}}(Q, L) = \langle D_1 f, L d_2 h\rangle - \langle D_1 h, L d_2 f\rangle + 2\langle L d_2 f, \mathcal{R}(Q)(L d_2 h)\rangle,$$

where the derivatives are evaluated at the point (Q, L).

Theorem 5. *The bi-Hamiltonian vector field V_k on $\mathfrak{M}$, given by*

$$V_k[F] =: \{F, H_k\}_2 = \{F, H_{k+1}\}_1, \qquad k \in \mathbb{N},$$

induces a derivation of $C^\infty(\mathbb{T}^n_{\text{reg}} \times \mathrm{i}\mathfrak{u}(n))^{\mathcal{N}(n)}$. Up to infinitesimal gauge transformations, this is given by the vector field W_k on $\mathbb{T}^n_{\text{reg}} \times \mathrm{i}\mathfrak{u}(n)$ that satisfies

$$\dot{Q}Q^{-1} := W_k[Q]Q^{-1} = (\mathrm{i}L^k)_{\text{diag}}, \quad \dot{L} := W_k[L] = [\mathcal{R}(Q)(\mathrm{i}L^k), L].$$

As derivations of $\mathcal{N}(n)$-invariant functions, $f = F \circ \iota$ and $h_k = H_k \circ \iota$, these reduced evolutional derivations obey

$$W_k[f] = \{f, h_k\}_2^{\text{red}} = \{f, h_{k+1}\}_1^{\text{red}}.$$

Summary: We have shown that Poisson reduction of the bi-Hamiltonian hierarchy of ‘free motion’ on $\mathfrak{M} = T^*\mathrm{U}(n)$ results in a bi-Hamiltonian hierarchy describing the time development of the gauge invariant observables forming $C^\infty(\mathbb{T}^n_{\text{reg}} \times \mathrm{i}\mathfrak{u}(n))^{\mathcal{N}(n)}$. The reduced hierarchy is called ‘trigonometric spin Ruijsenaars–Sutherland hierarchy’.

Interpretation as a spin Sutherland model (well-known): Introduce new variables by the diffeomorphism:

$$\mathbb{T}^n_{\mathrm{reg}} \times \mathrm{i}\mathfrak{u}(n) \ni (Q, L) \Longleftrightarrow (Q, p, \phi) \in \mathbb{T}^n_{\mathrm{reg}} \times \mathrm{i}\mathfrak{u}(n)_{\mathrm{diag}} \times \mathrm{i}\mathfrak{u}(n)_\perp$$

$$\text{using} \quad L(Q, p, \phi) := p + (\mathcal{R}(Q) + \frac{1}{2}\mathrm{id})(\phi).$$

The entries p_j of p and q_j in $Q_j = e^{\mathrm{i}q_j}$ form canonically conjugate pairs, and are combined with the Poisson algebra of the quotient

$\mathfrak{u}(n)^*//_0\mathbb{T}^n = (\mathrm{i}\mathfrak{u}(n)_\perp)/\mathbb{T}^n$. The space of physical observables becomes

$$C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathrm{i}\mathfrak{u}(n)_{\mathrm{diag}} \times \mathrm{i}\mathfrak{u}(n)_\perp)^{\mathcal{N}(n)},$$

and the reduced first Poisson bracket takes the form

$$\{\mathcal{F}, \mathcal{H}\}_1^{\mathrm{red}}(Q, p, \phi) = \langle D_Q\mathcal{F}, d_p\mathcal{H}\rangle - \langle D_Q\mathcal{H}, d_p\mathcal{F}\rangle + \langle \phi, [d_\phi\mathcal{F}, d_\phi\mathcal{H}]\rangle.$$

In these variables, we get the standard spin Sutherland Hamiltonian

$$\mathcal{H}_2(Q, p, \phi) := \frac{1}{2}(L(Q, p, \phi)^2) = \frac{1}{2}\sum_i p_i^2 + \frac{1}{8}\sum_{i\neq j} \frac{|\phi_{ij}|^2}{\sin^2 \frac{q_i - q_j}{2}}.$$

The spin variable ϕ can be restricted by fixing the values of the Casimir functions $C_i \in C^\infty(\mathfrak{u}(n)^*)^{\mathrm{U}(n)}$, and a special choice gives the spinless Sutherland model.

Interpretation as a spin Ruijsenaars model: Restrict attention to

$$\mathbb{T}^n_{\mathrm{reg}} \times \exp(\mathrm{i}\mathfrak{u}(n)) \subset \mathbb{T}^n_{\mathrm{reg}} \times \mathrm{i}\mathfrak{u}(n),$$

where L can be uniquely written in the form

$$L = e^p b_+ (b_+)^\dagger e^p \quad \text{with} \quad p \in \mathfrak{b}(n)_0,\ b_+ \in \exp(\mathfrak{b}(n)_+) =: \mathrm{B}(n)_+.$$

Then consider the invertible change of variables

$$(Q, L) \longleftrightarrow (Q, p, b_+) \longleftrightarrow (Q, p, \lambda(Q, b_+)) \text{ with } \lambda(Q, b_+) = b_+^{-1} Q^{-1} b_+ Q.$$

λ varies freely in the triangular nilpotent subgroup $\mathrm{B}(n)_+ < \mathrm{B}(n)$. This gives the identification

$$C^\infty\left(\mathbb{T}^n_{\mathrm{reg}} \times \exp(\mathrm{i}\mathfrak{u}(n))\right)^{\mathcal{N}(n)} \longleftrightarrow C^\infty\left(\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)_+\right)^{\mathcal{N}(n)}.$$

For $\mathcal{F}, \mathcal{H} \in C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)_+)^{\mathcal{N}(n)}$, the change of variables leads to the 'decoupled form' of the second Poisson bracket:

$$2\{\mathcal{F}, \mathcal{H}\}_2^{\mathrm{red}}(Q, p, \lambda) = \langle D_Q \mathcal{F}, d_p \mathcal{H} \rangle - \langle D_Q \mathcal{H}, d_p \mathcal{F} \rangle + \langle D'_\lambda \mathcal{F}, \lambda^{-1}(D_\lambda \mathcal{H})\lambda \rangle.$$

The last term encodes the natural Poisson bracket on $\mathrm{B}(n)//_0\mathbb{T}^n$, which is the Poisson-Lie analogue of $\mathfrak{u}(n)^*//_0\mathbb{T}^n$.

In terms of these variables, the main Hamitonian $\mathrm{tr}(L)$ has the form

$$\mathrm{tr}(L) = \sum_{i=1}^{n} e^{2p_i} V_i(Q,\lambda) \quad \text{with} \quad V_i(Q,\lambda) = \left(b_+(Q,\lambda) b_+(Q,\lambda)^\dagger\right)_{ii},$$

and thus the reduced system can be interperted as a spin RS model. The corresponding open subset of the reduced phase space is

$$\left(\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{b}(n)_0 \times (\mathrm{B}(n)_+/\mathbb{T}^n)\right) \Big/ S_n.$$

We obtain Poisson subspaces by restricting $\mathrm{B}(n)_+/\mathbb{T}^n$ to $\mathbb{T}^n$-reduced dressing orbits of $\mathrm{U}(n)$. The dressing orbits $\widetilde{\mathcal{O}} \subset \mathrm{B}(n)$ are obtained by fixing the Casimirs, $\mathcal{C}_i \in C^\infty(\mathrm{B}(n))^{\mathrm{U}(n)}$. The smallest non-trivial dressing orbit gives the standard *spinless*, *trigonometric* (real) RS model.

- Does the bi-Hamiltonian story generalize in a reasonable manner if we replace $\mathrm{U}(n)$ by an arbitrary compact simple Lie group?
- What about generalization to spin Sutherland and RS models of Gibbons–Hermsen and Krichever–Zabrodin type, and what about the elliptic case?
- Outstanding open question: **How to obtain the standard spinless, hyperbolic (real, repulsive) RS model by Hamiltonian reduction?**

A very incomplete list of references

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