The solutions of $\mathfrak{gl}_{m|n}$ Bethe ansatz equation and rational pseudodifferential operators

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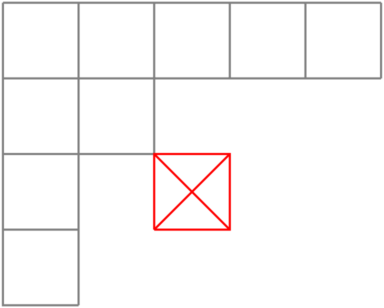
An $(m|n)$ parity sequence $\boldsymbol{s} = (s_1, \ldots, s_{m+n}), s_i \in \{\pm 1\}$, is a sequence such that 1 occurs exactly m times.

The Borel subalgebra of $\mathfrak{gl}_{m|n}$ with respect to $\boldsymbol{s}$ is $\mathfrak{b}_{\boldsymbol{s}}$. The positive simple roots with respect to $\boldsymbol{s}$ are $\alpha_i^{\boldsymbol{s}} = \epsilon_i^{\boldsymbol{s}} - \epsilon_{i+1}^{\boldsymbol{s}}, i = 1, \ldots, m+n-1$.

The polynomial $\mathfrak{gl}_{m|n}$-modules are parametrized by $(m|n)$-hook partition. The highest weight of a polynomial $\mathfrak{gl}_{m|n}$-module with respect to $\mathfrak{b}_{\boldsymbol{s}}$ can be found from the partition.



a $(2|2)$-hook parition

Given a sequence of $\mathfrak{gl}_{m|n}$-modules $(V_1,\ldots,V_k)$, a sequence of pairwise distinct complex numbers $\boldsymbol{z} = (z_1,\ldots,z_k)$, the (quadratic) Gaudin Hamiltonians $\mathcal{H}_r \in \mathrm{End}(\bigotimes_{r=1}^k V_r)$, $r = 1,\ldots,k$, are given by

$$\mathcal{H}_r = \sum_{\substack{r'=1 \\ r'\neq r}}^{k} \frac{\sum_{i,j=1}^{m+n} |j|\, e_{i,j}^{(r)} e_{j,i}^{(r')}}{z_r - z_{r'}}.$$

Lemma

1. *The Gaudin Hamiltonians mutually commute,* $[\mathcal{H}_r, \mathcal{H}'_r] = 0$*, for all* r, r'.
2. *The Gaudin Hamiltonians commute with the diagonal* $\mathfrak{gl}_{m|n}$ *action,* $[\mathcal{H}_r, X] = 0$*, for all* r *and all* $X \in \mathfrak{gl}_{m|n}$.
3. *If* V_r, $r = 1,\ldots,k$*, are polynomial modules, then for generic* z_r, $r = 1,\ldots,k$*, the Gaudin Hamiltonians are diagonalizable.*

The Bethe ansatz equation associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$, is a system of algebraic equations on variable $\boldsymbol{t}$:

$$\sum_{q=1}^{l_{i-1}} \frac{(\alpha^{s}_{i-1}, \alpha^{s}_{i})}{t^i_p - t^{i-1}_q} + \sum_{\substack{q=1\\ q\neq p}}^{l_i} \frac{(\alpha^{s}_{i}, \alpha^{s}_{i})}{t^i_p - t^{i}_q} + \sum_{q=1}^{l_{i+1}} \frac{(\alpha^{s}_{i+1}, \alpha^{s}_{i})}{t^i_p - t^{i+1}_q} = \sum_{r=1}^{k} \frac{(\lambda^{s}_{r}, \alpha^{s}_{i})}{t^i_p - z_r},$$

where $i = 1, \dots, m+n-1$, $p = 1, \dots, l_i$, see [MVY].

For i such that $s_i \neq s_{i+1}$, the BAEs related to t^i_p are the same for $p = 1, \dots, l_i$. Suppose t is the a solution of this equation of multiplicity a. If $\boldsymbol{t}$ is a solution of BAE, then we require the number of $t^i_p = t$ is at most a.

Define a sequence of polynomials $\boldsymbol{T^s} = (T_1^{\boldsymbol{s}}, \ldots, T_{m+n}^{\boldsymbol{s}})$ associated to $\boldsymbol{s}$, $\boldsymbol{z}$, and $\boldsymbol{\lambda}$,

$$T_i^{\boldsymbol{s}}(x) = \prod_{r=1}^{k} (x - z_r)^{s_i(\lambda_r^s, \epsilon_i^s)}, \ i = 1, \ldots, m+n.$$

Suppose $\boldsymbol{t}$ is a solution of BAE associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$, then define a sequence of polynomials $\boldsymbol{y} = (y_1, \ldots, y_{m+n-1})$ by

$$y_i(x) = \prod_{p=1}^{l_i} (x - t_p^i), \ i = 1, \ldots, m+n-1.$$

We say the sequence of polynomials $\boldsymbol{y}$ represents $\boldsymbol{t}$.

A sequence of polynomials $\boldsymbol{y}$ is generic with respect to $\boldsymbol{s}$, $\boldsymbol{z}$, and $\boldsymbol{\lambda}$, if
(1) if $s_i = s_{i+1}$, then $y_i(x)$ has only simple roots;
(2) y_i and $y_{i\pm 1}$ have no common roots;
(3) $y_i(x)$ and $T_i^{\boldsymbol{s}}(x)(T_{i+1}^{\boldsymbol{s}}(x))^{-s_i s_{i+1}}$ have no common roots.

Given $\boldsymbol{T^s}$, suppose $s_i \neq s_{i+1}$, then define a monic polynomial π_i^s which has only simple roots and $\pi_i^s(x) = 0$ if and only if $T_i^s T_{i+1}^s(x) = 0$.

Theorem

Let $\boldsymbol{y} = (y_1, \dots, y_{m+n-1})$ be a sequence of polynomials generic with respect to $\boldsymbol{s}$, $\boldsymbol{z}$, and $\boldsymbol{\lambda}$, such that $\deg y_i = l_i$, $i = 1, \dots, m+n-1$.

1. *The sequence $\boldsymbol{y}$ represents a solution of the BAE associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$, if and only if for each $i = 1, \dots, m+n-1$, there exists a polynomial $\widetilde{y}_i$, such that*

$$\begin{aligned} \mathrm{Wr}\left(y_i, \widetilde{y}_i\right) &= T_i^s \left(T_{i+1}^s\right)^{-1} y_{i-1} y_{i+1} && \text{if} \quad s_i = s_{i+1}, \\ y_i \, \widetilde{y}_i &= \ln' \left(\frac{T_i^s T_{i+1}^s y_{i-1}}{y_{i+1}} \right) \pi_i^s y_{i-1} y_{i+1} && \text{if} \quad s_i \neq s_{i+1}. \end{aligned}$$

2. *Let i be such that $\widetilde{y}_i \neq 0$. If $\boldsymbol{y}^{[i]} = (y_1, \dots, \widetilde{y}_i, \dots, y_{m+n-1})$ is generic with respect to $\boldsymbol{s}^{[i]} = (s_1, \dots, s_{i+1}, s_i, \dots, s_{m+n})$, $\boldsymbol{z}$, and $\boldsymbol{\lambda}$, then $\boldsymbol{y}^{[i]}$ represents a solution of the BAE associated to $\boldsymbol{s}^{[i]}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}^{[i]}$, where $\boldsymbol{l}^{[i]} = (l_1, \dots, \widetilde{l}_i, \dots, l_{m+n-1})$, $\widetilde{l}_i = \deg \widetilde{y}_i$.*

If $\boldsymbol{y}^{[i]}$ is generic with respect to $\boldsymbol{s}^{[i]}$, $\boldsymbol{\lambda}$, and $\boldsymbol{z}$, then by the above theorem, we can apply the reproduction procedure again.

The closure of the set of all pairs $(\widetilde{\boldsymbol{y}}, \widetilde{\boldsymbol{s}})$ obtained from the initial pair $(\boldsymbol{y}, \boldsymbol{s})$ by repeatedly applying all possible reproductions, $P_{(\boldsymbol{y},\boldsymbol{s})}$, is called the $\mathfrak{gl}_{m|n}$ population of solutions of the BAE associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$, originated at $\boldsymbol{y}$,

$$P_{(\boldsymbol{y},\boldsymbol{s})} \subset (\mathbb{P}(\mathbb{C}[x]))^{m+n-1} \times S_{m|n}.$$

By definition, $P_{(\boldsymbol{y},\boldsymbol{s})}$ decomposes as a disjoint union over parity sequences,

$$P_{(\boldsymbol{y},\boldsymbol{s})} = \bigsqcup_{\widetilde{\boldsymbol{s}} \in S_{m|n}} P^{\widetilde{\boldsymbol{s}}}_{(\boldsymbol{y},\boldsymbol{s})}, \qquad P^{\widetilde{\boldsymbol{s}}}_{(\boldsymbol{y},\boldsymbol{s})} = P_{(\boldsymbol{y},\boldsymbol{s})} \cap \left((\mathbb{P}(\mathbb{C}[x]))^{m+n-1} \times \{\widetilde{\boldsymbol{s}}\} \right).$$

The division ring of rational pseudodifferential operators $\mathbb{C}(x)(\partial)$ is the division subring of

$$\mathbb{C}(x)((\partial^{-1})) = \left\{ \sum_{r=-\infty}^{a} f_r \, \partial^r, \, f_r \in \mathbb{C}(x), \, a \in \mathbb{Z} \right\},$$

generated by $\mathbb{C}(x)[\partial]$, see [CDK].

Define a rational pseudodifferential operator $R^{s}(\boldsymbol{y}, \boldsymbol{T}^{s}) \in \mathbb{C}(x)(\partial)$,

$$R^{s}(\boldsymbol{y}, \boldsymbol{T}^{s}) = d_1^{s_1}(\boldsymbol{y}, \boldsymbol{T}^{s}) \dots d_{m+n}^{s_{m+n}}(\boldsymbol{y}, \boldsymbol{T}^{s}),$$

where $d_i(\boldsymbol{y}, \boldsymbol{T}^{s}) = \partial - s_i \ln' \frac{T_i^{s} y_{i-1}}{y_i}$.

Theorem

Let $\boldsymbol{y}$ represents a solution of BAE associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$. Then the rational pseudodifferential operator $R^{s}(\boldsymbol{y}, \boldsymbol{T}^{s})$ is invariant under reproduction procedure: $R^{s}(\boldsymbol{y}, \boldsymbol{T}^{s}) = R^{s^{[i]}}(\boldsymbol{y}^{[i]}, \boldsymbol{T}^{s^{[i]}})$.

When $\boldsymbol{\lambda}$ is a typical sequence of polynomial $\mathfrak{gl}_{m|n}$ weights, the operator $R_P^{\boldsymbol{s}_0} = D_{\bar{0}} D_{\bar{1}}^{-1}$ produces a vector superspace

$$W_P = \ker D_{\bar{0}} \bigoplus \ker D_{\bar{1}} \subset \mathbb{C}(x).$$

A full flag of a vector superspace W is called a full superflag if it is generated by a homogeneous basis. The set of all full superflags $\mathcal{F}(W)$ decomposes

$$\mathcal{F}(W) = \bigsqcup_{\boldsymbol{s} \in S_{m|n}} \mathcal{F}^{\boldsymbol{s}}(W),$$

where each $\mathcal{F}^{\boldsymbol{s}}(W)$ is isomorphic to $\mathcal{F}(W_{\bar{0}}) \times \mathcal{F}(W_{\bar{1}})$.

Theorem

Let $\boldsymbol{\lambda}$ be a typical sequence of polynomial $\mathfrak{gl}_{m|n}$ weights. The variety of superflags $\mathcal{F}(W_P)$ is canonically identified with the set of complete factorizations $\mathcal{F}(R_P)$ and the population P. Moreover, for each $\boldsymbol{s}$, we have $\mathcal{F}^{\boldsymbol{s}}(W_P) \cong \mathcal{F}^{\boldsymbol{s}}(R_P) \cong P^{\boldsymbol{s}}$.

Define

$$M = (\delta_{i,j}\partial - |i| e_{i,j}(x))_{i,j=1,\dots,m+n}.$$

The $\mathfrak{gl}_{m|n}$ Bethe algebra $\mathfrak{B} \subset U\mathfrak{gl}_{m|n}[t]$ is the subalgebra generated by $b_{a,r}$, see [MR], where $b_{a,r}$ are given by:

$$\mathrm{Ber}\, M = \mathrm{cdet}(M_{i,j})_{i,j=1,\dots,m} \cdot \mathrm{rdet}(M^{-1}_{m+i,m+j})_{i,j=1,\dots,n} = \sum_{r=-\infty}^{m-n} \sum_{a=-\infty}^{0} b_{a,r} x^a \partial^r.$$

Conjecture

Let $\boldsymbol{y}$ represent a solution of the BAE associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$. Then there exists a joint eigenvector $w(\boldsymbol{y}, \boldsymbol{T}^s)$ of $\mathfrak{B}$ in the singular space of $\mathrm{L}(\boldsymbol{\lambda})$ with respect to $\mathfrak{b}_s$ of weight $\lambda^{s,\infty}$. Moreover, the action of $\mathfrak{B}$ on $w(\boldsymbol{y}, \boldsymbol{T}^s)$ is given by

$$\mathrm{Ber}\, M\, w(\boldsymbol{y}, \boldsymbol{T}^s) = R^s(\boldsymbol{y}, \boldsymbol{T}^s)\, w(\boldsymbol{y}, \boldsymbol{T}^s).$$

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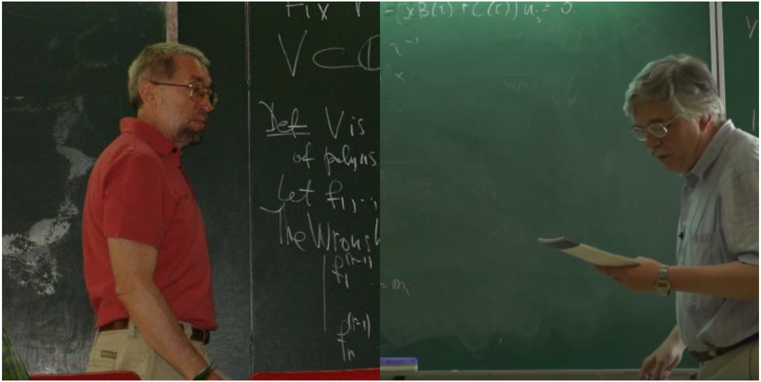
Thank You!



Vitaly Tarasov and Alexander Varchenko published their first joint work in 1994. Since then they have 52 publications.