Bethe subalgebras in Yangians

Aleksei Ilin

National Research University
Higher School of Economics
Faculty of Mathematics
Moscow, Russia

Zurich,
2019

J realization of the Yangian

Definition

The Yangian $Y(\mathfrak{g})$ is a unital associative algebra over $\mathbb{C}$ generated by the elements $\{x, J(x) \mid x \in \mathfrak{g}\}$ with the following defining relations:

$$xy - yx = [x, y], \qquad J([x, y]) = [J(x), y],$$

$$J(cx + dy) = cJ(x) + dJ(y),$$

$$[J(x), [J(y), z]] - [x, [J(y), J(z)]] = \sum_{\lambda,\mu,\nu\in\Lambda} \langle [x, x_\lambda], [[y, x_\mu], [z, x_\nu]] \rangle \{x_\lambda, x_\mu, x_\nu\},$$

$$[[J(x), J(y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]] = \sum_{\lambda,\mu,\nu\in\Lambda} \left(\langle [x, x_\lambda], [[y, x_\mu], [[z, w], x_\nu]] \rangle + \langle [z, x_\lambda], [[w, x_\mu], [[x, y], x_\nu]] \rangle \right) \{x_\lambda, x_\mu, J(x_\nu)\}$$

for all $x, y, z, w \in \mathfrak{g}$ and $c, d \in \mathbb{C}$, where $\langle \cdot, \cdot \rangle$ is a fixed non-degenerate invariant bilinear form on $\mathfrak{g}$, $\{x_\lambda\}_{\lambda\in\Lambda}$ is some orthonormal basis of $\mathfrak{g}$, $\{x_1, x_2, x_3\} = \frac{1}{24} \sum_{\pi\in\mathfrak{S}_3} x_{\pi(1)} x_{\pi(2)} x_{\pi(3)}$ for all $x_1, x_2, x_3 \in Y(\mathfrak{g})$.

Universal R-matrix

Definition

The universal matrix for the Yangian $Y(\mathfrak{g})$ is an element

$$\hat{R}(u) = Id + \sum_{k \geqslant 1} \hat{R}^{(k)} u^{-k} \in (Y(\mathfrak{g}) \otimes Y(\mathfrak{g}))[[u^{-1}]]$$

with the following properties:
1) $(\mathrm{Id} \otimes \Delta)\hat{R}(u) = \hat{R}_{12}(u)\hat{R}_{13}(u)$;
2) $\tau_{0,u}\Delta^{op}(X) = \hat{R}(u)^{-1}(\tau_{0,u}\Delta(X))\hat{R}(u)$ for all $X \in Y(\mathfrak{g})$.
Here τ_u is an automorphism of $Y(\mathfrak{g})$ such that $x \mapsto x, J(x) \mapsto J(x) + ux$ for all $x \in \mathfrak{g}$ and $\tau_{0,u} = \tau_0 \otimes \tau_u$.

Proposition

The universal R-matrix is unique.

RTT realization

Let (ρ, V) be any non-trivial representation of $Y(\mathfrak{g})$. Let $R(u) = (\rho \otimes \rho)\hat{R}(-u)$. We fix a basis of V and regard $R(u) \in \mathrm{End}\,(V)^{\otimes 2}[[u^{-1}]]$ as a matrix in this basis.

Definition

The Yangian $Y_V(\mathfrak{g})$ is a unital associative algebra generated by the elements $t_{ij}^{(r)}, 1 \leqslant i, j \leqslant \dim V; r \geqslant 1$ with the defining relations

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v) \text{ in } \mathrm{End}\,(V)^{\otimes 2} \otimes Y_V(\mathfrak{g})[[u^{-1}, v^{-1}]],$$

$$S^2(T(u)) = T(u + \frac{1}{2}c_{\mathfrak{g}}),$$

where $S(T(u)) = T(u)^{-1}$ is the antipode map and $c_{\mathfrak{g}}$ is the value of the Casimir element of $\mathfrak{g}$ on the adjoint representation. Here

$$T(u) = [t_{ij}(u)]_{i,j=1,\ldots,\dim V} \in \mathrm{End}\, V \otimes Y_V(\mathfrak{g}),$$

$$t_{ij}(u) = \delta_{ij} + \sum_r t_{ij}^{(r)} u^{-r}$$

and $T_1(u)$ (resp. $T_2(u)$) is the image of $T(u)$ in the first (resp. second) copy of $\mathrm{End}\, V$.

Isomorphism between J and RTT realization

Theorem (V. Drinfeld, C. Wendlandt)

The map $\psi : Y_V(\mathfrak{g}) \to Y(\mathfrak{g})$ such that

$$T(u) \mapsto (\rho \otimes 1)\hat{R}(-u).$$

is an isomorphism.

From know we consider $V = \bigoplus_{i=1}^{n} V(\omega_i, 0)$ sum of fundamental representations of $Y(\mathfrak{g})$. Note that the restriction of $V(\omega_i, 0)$ to $\mathfrak{g}$ decomposes as

$$V(\omega_i, 0) = V_{\omega_i} \oplus \bigoplus_{\mu < \omega_i} V_\mu^{\oplus k_\mu}$$

Here V_μ is the irreducible representation of $\mathfrak{g}$ of highest weight μ and $\mu < \omega_i$ means that $\omega_i - \mu$ is a sum of positive roots, $k_\mu \in \mathbb{Z}_{\geqslant 0}$.

Definition of Bethe subalgebras

Let $\rho_i : Y(\mathfrak{g}) \to \operatorname{End} V(\omega_i, 0)$ be the i-th fundamental representation of $Y(\mathfrak{g})$. Let

$$\pi_i : V \to V(\omega_i, 0)$$

be the projection. Let $T^i(u) = \pi_i T(u) \pi_i$ be the submatrix of $T(u)$-matrix, corresponding to i-th fundamental representation. Let $\tilde{G}$ be the simply connected Lie group, corresponding to the Lie algebra $\mathfrak{g}$.

Definition

Let $C \in \tilde{G}$. Bethe subalgebra $B(C) \subset Y_V(\mathfrak{g})$ is the subalgebra generated by all Fourier coefficients of the following series with the coefficients in $Y_V(\mathfrak{g})$

$$\tau_i(u, C) = \operatorname{tr}_{V(\omega_i,0)} \rho_i(C) T^i(u), \quad 1 \leqslant i \leqslant n.$$

Let G be the adjoint Lie group corresponding to the Lie algebra $\mathfrak{g}$. In fact, Bethe subalgebras is parameterized by G.

Bethe subalgebras

Let G^{reg} be the set of regular elements of G, T – maximal torus, T^{reg} – the set of regular elements of torus.

Theorem

1) For any $C \in G^{reg}$ Bethe subalgebra $B(C)$ is a free polynomial algebra and the coefficients of $\tau_i(u, C)$ are free generators of $B(C)$.
2) For any $C \in T^{reg}$ Bethe subalgebra $B(C)$ is a maximal commutative subalgebra of $Y_V(\mathfrak{g})$.

Corollary

For any $C \in T^{reg}$ Bethe subalgebra $B(C)$ in $Y(\mathfrak{g})$ is generated by the coefficients of

$$\operatorname{tr}_V \rho(C)(\rho \otimes 1)\hat{R}(u),$$

where (ρ, V) are all finite-dimensional representation of $Y(\mathfrak{g})$.

Let V be a representation of $\tilde{G}$. This defines the map

$$G \to \mathbb{P}(\operatorname{End} V).$$

If $V = V_\lambda \oplus \bigoplus_{\mu<\lambda} V_\mu^{\oplus k_\mu}$, $k_m u \geqslant 0$ and V_λ is irreducible of regular highest weight λ, then the closure of the image of G in $\mathbb{P}(\operatorname{End} V)$ is a smooth projective variety called De-Concini - Procesi wonderful compactification $\overline{G}$.

We consider the closure of G in $\prod_i \mathbb{P}(\mathrm{End}\,(V(\omega_i,0)))$. In is known that it is isomorphic to $\overline{G}$.
Suppose that $X = (X_1, \ldots, X_n) \in \overline{G} \subset \prod_i \mathbb{P}(\mathrm{End}\,(V(\omega_i,0)))$. We define the subalgebra $B(X)$ of $Y_V(\mathfrak{g})$ using the same formulas just changing $\rho_i(C)$ to X_i:

$$\tau_i(u, C) = \mathrm{tr}\,_{V(\omega_i,0)} X_i T^i(u), \quad 1 \leqslant i \leqslant n.$$