The Bethe ansatz equations and integrable system of particles

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Motivation

- A. Varchenko, T. Woodruff, *Critical points of master functions and the mKdV hierarchy of type $A_{2n}^{(2)}$*, Representations of Lie algebras, quantum groups and related topics, 205–233, Contemp. Math., **713**, Amer. Math. Soc., Providence, RI, 2018
- A. Varchenko, T. Woodruff, *Critical points of master functions and mKdV hierarchy of type $C_n^{(1)}$*, 1–31
- A. Varchenko, D. Wright, *Critical points of master functions and integrable hierarchies*, Advances in Mathematics 263 (2014) 178–229.
- A. Varchenko, T. Woodruff, D. Wright, *Critical points of master functions and the mKdV hierarchy of type $A_2^{(2)}$*, in Bridging Algebra, Geometry, and Topology, Springer Proceedings in Mathematics and Statistics, vol. **96**, 167–195, Springer, 2014

For the affine Lie algebra $\widehat{\mathfrak{sl}_N}$ and its trivial representation the associated system of the Bethe ansatz equations has the form

$$\sum_{i'\neq i} \frac{2}{u_i^{(n)} - u_{i'}^{(n)}} - \sum_{i=1}^{k_{n+1}} \frac{1}{u_i^{(n)} - u_{i'}^{(n+1)}} - \sum_{i=1}^{k_{n-1}} \frac{1}{u_i^{(n)} - u_{i'}^{(n-1)}} = 0,$$

for $n = 1, \ldots, N$ and $i = 1, \ldots, k_n$. Here $k_{N+n} = k_n$ and $u_i^{(N+n)} = u_i^{(n)}$ for all i, n. The system itself depends on the choice of nonnegative integers $k_1, \ldots, k_N$, which must satisfy the equation

$$\sum_{j=1}^{N} \frac{(k_j - k_{j+1})^2}{2} - \sum_{j=1}^{N} k_j = 0\,.$$

- *The set of solutions of such a system forms one cell invariant under the action of commuting flows the N mKdV integrable hierarchy*

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The original motivation was to extend these results to the $\widehat{\mathfrak{sl}_N}$ XXX quantum integrable model, associated with the trivial representation of $\widehat{\mathfrak{sl}_N}$. In this case the Bethe ansatz equations take the form

$$\frac{y_{n-1}(u_j^{(n)}+1)y_n(u_j^{(n)}-1)y_{n+1}(u_j^{(n)})}{y_{n-1}(u_j^{(n)})y_n(u_j^{(n)}+1)y_{n+1}(u_j^{(n)}-1)} = -1,$$

where

$$y_n(x) = \prod_{i=1}^{k_n}(x - u_i^{(n)})$$

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Solution

Given integer ν and $(N+\nu) \times (N+\nu)$ matrix W such that its upper-right $\nu \times \nu$ corner U is *nilpotent*,

$$W = \begin{pmatrix} V & U \\ * & * \end{pmatrix} \qquad \text{and} \qquad U^r = 0 \quad \text{for some} \quad r < \nu\,.$$

we define $\nu \times N\nu$ matrix Q

$$Q = \begin{pmatrix} V & UV & U^2 V & U^3 V & \cdots \end{pmatrix}$$

and then $(N+\nu) \times N(\nu+1)$-matrix P

$$P = \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & Q \end{pmatrix},$$

and the matrix A

$$A = WP\,.$$

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Introduce the polynomials

$$f_k(x,t) = \sum_{j=0} a_{k,j}\, \chi_j(x,t), \qquad k = 1, \ldots, N+\nu\,.$$

where the polynomials $\chi_n(x,t)$ are defined by the formula

$$(1+z)^x e^{\sum_{j=1}^{\infty} t_j z^j} = \sum_{n=0}^{\infty} \chi_n(x,t) z^n$$

Theorem

The polynomials $(y_0, \ldots, y_N)$ defined by the formula

$$y_n(x,t) = \det(f_i(x+j,t)), i,j = 0, \ldots, \nu + n$$

extends to periodic solution of the BA equations. All solution of the N-periodic BA equations are given by this formula with $t = 0$.

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Rational solutions of integrable PDE $\Leftrightarrow$ Integrable systems of particles

- Dependence of poles of the rational solutions of the KdV equations coincides with dynamics of rational Calogero-Moser system with respect to the flow generated by the Hamiltonian H_3 restricted to the stationary points of the flow corresponding to H_2 Hamiltonian (Airault, McKean, Moser, 1977)
- The theories of rational (trigonometric, elliptic) CM system and the theory of rational (trigonometric, elliptic) solutions of the KP equations are isomorphic (Kr; Choodnovski,Choodnovski,1978)

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Generating linear problem scheme (Kr)

Question: when a linear equation with rational coefficients has rational solutions ?

Examples. The basic auxiliary linear problems for the KP equation, $2D$ Toda

$$(A)\quad \partial_t\psi(x,t) = \partial_x^2\psi(x,t) + u(x,t)\psi(x,t), \quad u = 2\partial_x^2 \ln y(x,t)$$

$$(B)\quad \partial_t\psi(x,t) = \psi(x+1,t) + w(x,t)\psi(x,t,z), w = \partial_t \ln \frac{y(x+1,t)}{y(x,t)}$$

with

$$y(x,t) = \prod_{i=1}^{k}(x - u_i(t))$$

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Answer:

- (A) The Calogero-Moser system (Kr)

$$\ddot{u}_i = 2\sum_{j\neq i} \frac{1}{(u_i - u_j)^3}$$

- (B) The Ruijsenaars-Schneider system (Zabrodin-Kr)

$$\ddot{u}_i = \sum_{j\neq i} \dot{u}_i \dot{u}_j \left(\frac{1}{u_i - u_j - 1} + \frac{1}{u_i - u_j + 1} - \frac{2}{u_i - u_j} \right)$$

Generating linear problem

Lemma (Kr, Lipan,Wiegmann,Zabrodin)

The system of linear equations

$$\psi_{n+1}(x) = \psi_n(x+1) - v_n(x)\psi_n(x),$$

with

$$v_n(x) = \frac{y_n(x)y_{n+1}(x+1)}{y_n(x+1)y_{n+1}(x)},$$

where $(y_n(x))$ *is a given sequence of polynomials has a solution* $(\psi_n(x))$ *rational in* x *with the poles of* $\psi_n(x)$ *only at the zeros of* $y_n(x)$*, if and only if the zeros* $(u_i^{(n)})$ *of* $y_n(x)$ *satisfy the Bethe ansatz equation.*

Lemma

Let $y_n(x)$ be a sequence of polynomials (non-necessary periodic) whose roots satisfy the BA equations. Then

$$\psi_n(x,z) = z^n(1+z)^x \frac{\det \widehat{L}^{(n)}(x,z)}{\det L^{(n)}(z)}\,. \tag{1}$$

is a solutions of the generating problem.
Here

$$L^{(n)}(z) = (1+z)E - L(\gamma^{(n)}, u^{(n)})\,;$$

$$L(\gamma, u) := \frac{\gamma_i}{u_i - u_j - 1}$$

$$\gamma_i^{(n)} := \mathrm{Res}_{x=u_i^{(n)}-1} \frac{y_n(x)\, y_{n+1}(x+1)}{y_n(x+1)\, y_{n+1}(x)}$$

and $\widehat{L}^{(n)}(x,z)$ is $(k_n+1)\times(k_n+1)$ matrix with entries

$$\widehat{L}^{(n)}_{0,0}=1\,,\qquad \widehat{L}^{(n)}_{0,j}=\frac{1}{x-u_j^{(n)}}\,,\qquad \widehat{L}^{(n)}_{i,0}=-\gamma_i^{(n)}$$

$$\widehat{L}^{(n)}_{i,j}\;=\;L^{(n)}_{i,j}\,,\qquad i,j=1,\ldots,k_n\,.$$

$\Rightarrow$ *For each n the function $\Psi_n(x,z)$ is the Baker-Akhiezer function of k_n particle rational Ruijesennars-Schneider (RS) system* and $\widehat{L}^{(n)}(x,z)$ is $(k_n+1)\times(k_n+1)$ matrix with entries

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Rational RS system

The rational RS system with k particles is a Hamiltonian system with the Hamiltonian

$$H(u,p) = \sum_{i=1}^{k} \gamma_i, \quad \gamma_i := e^{p_i} \prod_{j \neq i} \left(\frac{(u_i - u_j - 1)(u_i - u_j + 1)}{(u_i - u_j)^2} \right)^{1/2}.$$

It is a completely integrable Hamiltonian system, whose equations of motion,

$$\dot{u}_i = \gamma_i, \;\; \dot{\gamma}_i = \sum_{j \neq i} \gamma_i \gamma_j \left(\frac{1}{u_i - u_j - 1} + \frac{1}{u_i - u_j + 1} - \frac{2}{u_i - u_j} \right),$$

admit the Lax representation $\dot{L} = [M, L]$ with

$$L_{ij}(u, \gamma) = \frac{\gamma_i}{u_i - u_j - 1}.$$

Direct spectral transform for the rational RS system

A point (u, γ) of the phase space of k-particle RS system defines the function

$$\Psi(x, z) = \det \widehat{L}(x, z)$$

The correspondence which assigns to a point (u, γ) a certain data characterizing analytic properties of Ψ in *the spectral parameter z* usually referred to as *direct spectral transform*.

Let $(\mu_i = \mu_i(u,\gamma))_{i=1}^q$ be the set of all distinct eigenvalues of $L(u,\gamma)$ of multiplicities $(m_i)_{i=1}^q$, i.e.

$$\det L(z \,|\, u,\gamma) = \prod_{i=1}^{q} (z - \mu_i + 1)^{m_i}\,, \qquad \mu_i \neq \mu_j\,.$$

Theorem

Let $(u,\gamma) \in \mathcal{P}_k$. Then for $j = 1,\ldots,q$, there is a unique m_j-dimensional vector subspace $W_j(u,\gamma)$ in the space of polynomials of degree $2m_j$ such that

$$\mathrm{Res}_{z=\mu_j-1} \frac{g(z)\Psi(x,z)}{(z-\mu_j+1)^{2m_j}} \;=\; 0\,, \qquad \forall\, g(x) \in W_j(u,\gamma)\,. \qquad (2)$$

The correspondence

$$(u,\gamma) \longmapsto (\mu, W)$$

is one-to-one with the open set of (μ, W).

Inverse spectral transform

Lemma

Given (μ, m, W) *there is a unique function* $\Psi(x, t, z)$*,*

$$\Psi(x,t,z) = (z+1)^x e^{\sum_{j=1}^{\infty} t_j z^j} \left(z^k + \sum_{s=1}^{k} \xi_\ell(x,t) z^{k-s} \right),$$

such that equations (2) hold.

The proof is by explicit construction. Choose a basis $g_{j,k}(z)$ in W_j. Then equations (2) can be represented in the form of the inhomogeneous linear system of equations

$$M(x,t \,|\, \mu, m, W)\, \xi(x,t) \; = \; -e_0, e_0 = (1,1,\ldots,1)^T$$

with some matrix M, whose entries are explicit expressions linear in the coefficients of the polynomials $g_{j,k}(z)$.

Inverse spectral transform

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The function Ψ can be written in the same determinant form as in (1):

$$\Psi(x, t, z \,|\, \mu, m, W) = \frac{\det \widehat{M}(x, t, z \,|\, \mu, m, W)}{y(x, t \,|\, \mu, m, W)} \,,$$

with

$$y(x, t \,|\, \mu, m, W) = \det M(x, t \,|\, \mu, m, W) \,.$$

Theorem

If $(y_n(x))$ represents a solutions of Bethe ansatz equations, then:

- *the eigenvalues $\mu_j^{(n)} \neq 1$ of $L(u^{(n)}, \gamma^{(n)})$ and the corresponding subspaces $W_j^{(n)}$ do not depend on n*
- *for the subspace $W_0^{(n)}$ corresponding to $\mu_0^{(n)} = 1$ the following statements*

$$W_0^{(n)} \subset W_0^{(n+1)}, \ \ \dim W_0^{(n+1)}/W_0^{(n)} = 1$$

  hold.

If $(y_n(x))$ represents a solutions of N-periodic Bethe ansatz equations, then $L(u^{(n)}, \gamma^{(n)})$ has only one eigenvalue $\mu = 1$ (of multiplicity k_n).

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If $(y_n(x))$ represents a solutions of N-periodic Bethe ansatz equations, then $L(u^{(n)}, \gamma^{(n)})$ has only one eigenvalue $\mu = 1$ (of multiplicity k_n).

Theorem

Let $y_n(x)$ be a generic sequence of polynomials of degrees k_n representing solution of the N-periodic Bethe ansatz equations. The correspondence

$$(y_n) \longmapsto (u^{(n)}, \gamma^{(n)}) \tag{3}$$

where

$$\gamma_i^{(n)} := \mathrm{Res}_{x=u_i^{(n)}-1} \frac{y_n(x)\, y_{n+1}(x+1)}{y_n(x+1)\, y_{n+1}(x)}$$

is an embedding of the space of solutions of the Bethe ansatz equation into the product of phase spaces of k_n-particle RS system, $n = 1, \dots, k_N$.
The image of this map is invariant under the hierarchy of the RS system (acting diagonally on the product of the phase spaces)

$$\partial_m u_i = \mathrm{Res}_{u_i} h_{m,m}(x)$$

where the polynomials $h_{s,m}(x)$ are defined recurrently by the formula

$$h_{s,m}(x) = \sum_{i=1}^{k} \left(\frac{(L^{s-1}\gamma)_i}{x-u_i} - \frac{(L^{s-1}\gamma)_i}{x-u_i+m} - \sum_{\ell=1}^{s-1} h_{\ell,m}(x) \frac{(L^{s-1-\ell}\gamma)_i}{x-u_i+m-\ell} \right)$$

Critical points of the Master function revisited revisited

Theorem

Let $y_n(x)$ be a generic sequence of polynomials of degrees k_n representing solution of the Bethe ansatz equations for the affine Lie algebra $\widehat{\mathfrak{sl}_N}$. The correspondence

$$(y_n) \longmapsto (u^{(n)}, p^{(n)}) \tag{4}$$

where

$$p_i^{(n)} := \sum_{j \neq i} \frac{1}{u_i^{(n)} - u_j^{(n)}} - \sum_{\ell \neq i} \frac{1}{u_i^{(n)} - u_\ell^{(n+1)}}$$

is an embedding of the space of solutions of the Bethe ansatz equation into the product of phase spaces of k_n-particle CM system, $n = 1, \ldots, k_N$.
The image of this map is invariant under the hierarchy of the CM system (acting diagonally on the product of the phase spaces)

The generating problem II

Lemma

The system of linear equations

$$\psi_n(x+1) - \psi_n(x-1) = w_n(x)\psi_{n+1}(x),$$

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$$w_n(x) = \frac{y_{n-1}(x)y_{n+1}(x)}{y_n(x+1)y_n(x-1)},$$

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$$\frac{y_{n-1}(u_j^{(n)}+1)y_n(u_j^{(n)}-2)y_{n+1}(u_j^{(n)}+1)}{y_{n-1}(u_j^{(n)-1})y_n(u_j^{(n)}+2)y_{n+1}(u_j^{(n)}-1)} = -1,$$

Welter's trisecant conjecture

Theorem

An indecomposable, principally polarized abelian variety (X, θ) is the Jacobian of a smooth curve of genus g if and only if there exist non-zero g-dimensional vectors $U \neq V (\mathrm{mod} \Lambda)$ such that the equation

$$\frac{\theta(Z+U)\,\theta(Z-V)\;\theta(Z-U+V)}{\theta(Z-U)\,\theta(Z+V)\;\theta(Z+U-V)} = -1$$

is valid on the theta-divisor $\Theta = \{Z \in X \mid \theta(Z) = 0\}$.

Happy birthday Vitalii and Sasha !