Equivariant $\mathcal{D}$-modules on varieties with finitely many orbits

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Outline

1. Equivariant $\mathcal{D}$-modules
2. Example: the space of $m \times n$ matrices
3. Spherical varieties
4. Local cohomology

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- Differentiating the action of G on X gives vector fields on X, so a map $\mathfrak{g} \to \Gamma(X, \mathcal{D}_X)$. When X is affine, equivariance of a $\mathcal{D}_X$-module means that the action of $\mathfrak{g}$ via $\mathfrak{g} \to \mathcal{D}_X$ can be integrated to an algebraic G-action.

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- $\mathcal{O}_X$ is equivariant, but $\mathcal{D}_X$ is not!
- Let $\text{mod}_G(\mathcal{D}_X)$ denote the full subcategory of equivariant $\mathcal{D}$-modules. It is closed under subquotients.

The category $\text{mod}_G(\mathcal{D}_X)$ of equivariant $\mathcal{D}$-modules

Now we consider the situation when G acts on X with *finitely many* orbits $O_0, O_1, \dots O_n$, where $\overline{O_n} = X$.

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- For each orbit $O \cong G/H$, we have $\mathrm{mod}_G(\mathcal{D}_O) \cong \mathrm{Rep}(H/H^0)$ (here H/H^0 is the component group of O).

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- Hence, there are finitely many simples in $\text{mod}_G(\mathcal{D}_X)$, parametrized by (O_p, V), with $0 \leq p \leq n$ and V an irrep. of the component group of O_p.
- The category $\text{mod}_G(\mathcal{D}_X)$ is equivalent to the category of finite-dimensional representations of a quiver (with relations) [Vilonen '94] [L., Walther '19].

The case of $m \times n$ matrices

Take $m \geq n \geq 1$ and let $X = \mathbb{C}^{m \times n}$ be space of $m \times n$ matrices, equipped with the action of the $G = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$.

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- We have the simples $D_0, D_1, \ldots, D_n$ in $\mathrm{mod}_G(\mathcal{D}_X)$ corresponding to orbits (all stabilizers are connected). Here $D_0 = \mathcal{D}_X/(x_1, \ldots, x_d) =: E$ and $D_n = \mathcal{O}_X =: S$.

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- When $m \neq n$, then the category $\mathrm{mod}_G(\mathcal{D}_X)$ is semi-simple.

The square case

When $m = n$, the roots of the Bernstein-Sato polynomial of the determinant give a filtration in $\text{mod}_G(\mathcal{D}_X)$:

$$0 \subsetneq S \subsetneq \langle \det^{-1} \rangle_{\mathcal{D}} \subsetneq \cdots \subsetneq \langle \det^{-n} \rangle_{\mathcal{D}} = S_{\det}$$

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$$\widehat{AA}_n : \quad (0) \rightleftarrows (1) \rightleftarrows \cdots \rightleftarrows (n-1) \rightleftarrows (n) \ ,$$

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$\widehat{AA}_n$ has finitely many indecomposable representations!

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Let G be a complex reductive group and B a Borel subgroup. We say X is a spherical variety, if B acts on X with finitely many orbits.

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Theorem (L., Walther '19)

Let X be a spherical variety of G, and $\mathcal{M}$ a G-equivariant simple $\mathcal{D}$-module. Then $\Gamma(X, \mathcal{M})$ has a multiplicity-free decomposition into irreducible G-modules (i.e. an irreducible G-module appears at most once). Moreover, if $\Gamma(X, \mathcal{M}) \neq 0$ then the characteristic cycle of $\mathcal{M}$ is also multiplicity-free.

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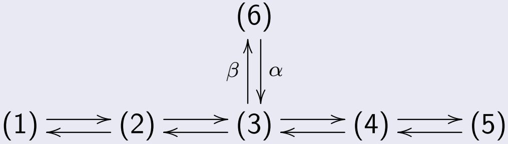
Some formulas for characters of equivariant $\mathcal{D}$-modules are calculated (for some non-spherical representations as well).

A classification result

The irreducible spherical representations have been classified by [Sato-Kimura '77] and [Kac '80].

Theorem (L., Walther '19)

Let X an irreducible G-spherical representation. Then $\operatorname{mod}_G(\mathcal{D}_X)$ *is given by a disjoint union of quivers of type* $\widehat{AA}_n$*, except in one case, when* $X = \mathbb{C}^{4\times 4}$ *and* $G = \mathrm{Sp}_4 \times \mathrm{GL}_4$*, when the quiver is*



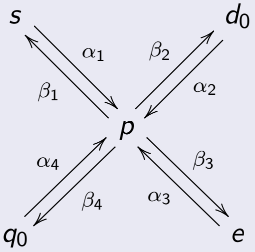
with all 2-cycles zero, and all compositions with the arrows α *or* β *are zero.*

A non-spherical example: binary cubic forms

$X = \operatorname{Sym}^3 \mathbb{C}^2$, $G = \mathrm{GL}_2(\mathbb{C})$. There are only 4 orbits, but 14 simple equivariant $\mathcal{D}$-modules (stabilizers not connected).

Theorem (L., Raicu, Weyman '19)

The quiver of the category $\operatorname{mod}_G(\mathcal{D}_X)$ *has a connected component*



with relations given by all 2-cycles and all non-diagonal compositions of two arrows.

An application: Local cohomology

Let Z be subvariety of X, and $\mathcal{M}$ any $\mathcal{O}_X$-module. $\mathcal{H}^0_Z(\mathcal{M}) =$ sheaf of sections of M with support in Z.

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$\mathcal{H}^0_Z(-)$ is left exact; consider its right derived functors $\mathcal{H}^i_Z(-)$ for $i \geq 0$.

If M is a $\mathcal{D}$-module, then so is $\mathcal{H}^i_Z(M)$. If moreover Z is G-stable and $\mathcal{M}$ is equivariant, then so is $\mathcal{H}^i_Z(\mathcal{M})$.

A general goal: Describe the $\mathcal{D}$-modules $\mathcal{H}^i_Z(\mathcal{O}_X)$ for any $i \geq 0$.

Example: back to matrices

$X = \mathbb{C}^{m \times n}$ be space of $m \times n$ matrices, equipped with the action of the $G = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, and $O_i =$ set of matrices of rank i.

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When $m \neq n$, the category $\mathrm{mod}_G(\mathcal{D}_X)$ is semi-simple, so each $H^j_{\overline{O}_t}(D_p)$ is a direct sum of $D_0, \dots, D_n$ (formula in [L., Raicu '18]).

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In the square case $m = n$, the indecomposables of main interest:

$$Q_p := \frac{S_{\det}}{\langle \det^{p+1-n} \rangle} \in \mathrm{mod}_G(\mathcal{D}_X) \text{ corresponds in } \mathrm{rep}(\widehat{AA}_n) \text{ to}$$

$$\mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \cdots \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0 \underset{0}{\overset{0}{\rightleftarrows}} \cdots \underset{0}{\overset{0}{\rightleftarrows}} 0 \quad (p \text{ 1's})$$

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Let $\mathrm{add}(Q)$ denote the subcategory of $\mathrm{mod}_G(\mathcal{D}_X)$ formed of $\mathcal{D}$-modules that are direct sums of $Q_0, Q_1, \dots Q_{n-1}$.

Direct sum decomposition in square case

$$q\text{-binomial: } \binom{a}{b}_q = \frac{(1-q^a)\cdot(1-q^{a-1})\cdots(1-q^{a-b+1})}{(1-q^b)\cdot(1-q^{b-1})\cdots(1-q)}$$

Theorem (L., Raicu '18)

We have that $H^j_{\overline{O}_t}(D_p) \in \operatorname{add}(Q)$ *(with* $t < p$*). Explicitly:*

$$\sum_{j\geq 0} [H^j_{\overline{O}_t}(D_p)] \cdot q^j = \sum_{s=0}^{t} [Q_s] \cdot q^{(p-t)^2} \cdot m_s(q^2),$$

where $m_t(q) = \binom{n-t}{p-t}_q$*, and for* $s = 0, \cdots, t-1$

$$m_s(q) = \binom{n-s}{p-s}_q \cdot \binom{p-1-s}{t-s}_q - \binom{n-s-1}{p-s-1}_q \cdot \binom{p-2-s}{t-1-s}_q$$

We also show that $H^j_{\overline{O}_t}(Q_p) \in \text{add}(Q)$ and give an explicit formula.

Hence, we can calculate all iterations $H^{i_1}_{\overline{O}_{t_1}}(H^{i_2}_{\overline{O}_{t_2}}(\cdots H^{i_r}_{\overline{O}_{t_r}}(D_p)\cdots))$

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In particular, we determine the *Lyubeznik numbers* $\lambda_{i,j}(\overline{O}_p)$ of determinantal varieties, given by

$$H^i_{\{0\}} H^{mn-j}_{\overline{O}_p}(S) = E^{\oplus \lambda_{i,j}(\overline{O}_p)}$$

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The Lyubeznik numbers are truly invariants of the (projective) determinantal varieties themselves, i.e. they do not depend on the choice of embedding into the projective space.

Lyubeznik numbers in the square case

Theorem (L., Raicu '18)

We have $\sum \lambda_{i,j}(\overline{O}_{n-1}) \cdot q^i \cdot w^j = (q \cdot w)^{n^2-1}$ *and for* $0 \leq p \leq n-2$ *we have*

$$\sum_{i,j \geq 0} \lambda_{i,j}(\overline{O}_p) \cdot q^i \cdot w^j =$$

$$= \sum_{s=0}^{p} q^{s^2+2s} \cdot \binom{n-1}{s}_{q^2} \cdot w^{p^2+2p+s\cdot(2n-2p-2)} \cdot \binom{n-2-s}{p-s}_{w^2}$$

Similar methods were applied to describe local cohomology and Lyubeznik numbers for other spaces of interest.