Recovering a linear problem from a nonlinear problem

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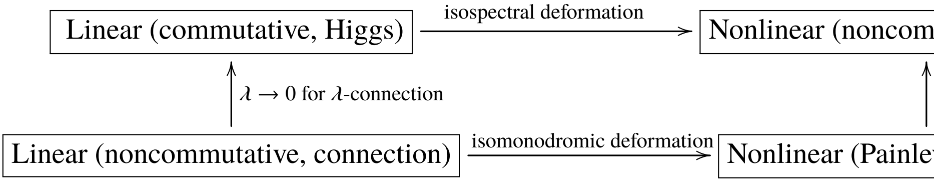
Linear to nonlinear

We consider the Painlevé-type equations or their autonomous (=isospectral) version, the Hitchin systems.
The direction “linear to nonlinear" is well-studied.



Jimbo-Miwa-Ueno [4], Inaba-Iwasaki-Saito [3], Rains, “Generalized Hitchin systems on rational surfaces"[13], “The birational geometry of noncommutative surfaces", [14], and more...

Nonlinear to linear?

What if we do not know linear problem in advance, and only have nonlinear integrable systems? Can we recover a linear problem?

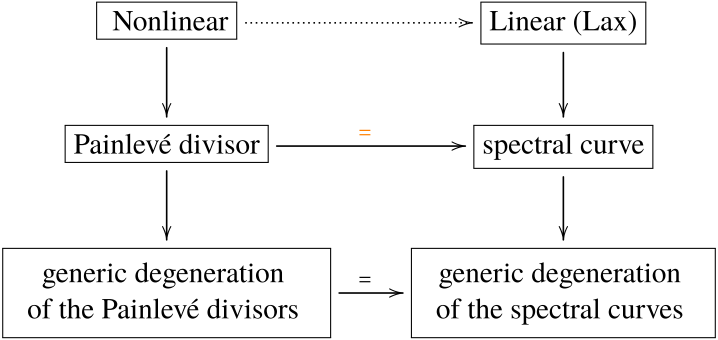
$$\boxed{\textbf{Linear}\text{ (Lax)}} \underset{?}{\overset{\text{Isospectral,Isomonodromic}}{\rightleftharpoons}} \boxed{\textbf{Nonlinear}\text{ (Hitchin, Painlevé)}}$$

Goal

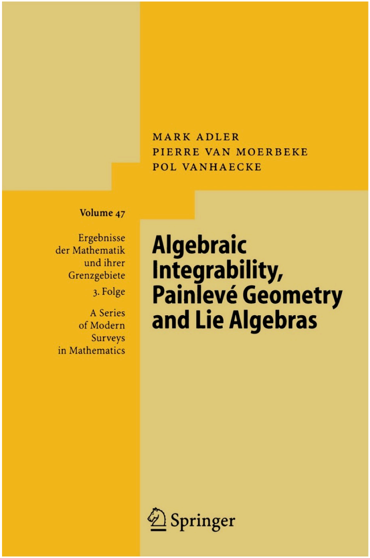
We will suggest a way to **recover a linear problem from a nonlinear problem** for the 4-dimensional autonomous Painlevé-type systems.

Summary

For the autonomous 4-dimensional Painlevé-type equations, we can recover a linear problem from a nonlinear problem (in principle).



Extracting geometrical data from nonlinear systems



Adler, Van Moerbeke, Vanhaecke, “Algebraic Integrablility, Painlevé Geometry, and Lie Algebras"[2]

- Kowalewski-Painlevé analysis
- compactification of the Liouville tori
- proving algebraic complete integrability
- etc.

Proposition 1 (Yoshida [16], Adler-van Moerbeke-Vanhake [2])

Suppose that V is a weight homogeneous vector field on $\mathbb{C}^n$, given by

$$\dot{x}_i = f_i(x_1, \ldots, x_n),\ (i = 1, \ldots, n)$$

and suppose that

$$x_i(t) = \sum_{k=0}^{\infty} x_i^{(k)} t^{-\nu_i + k},\ (i = 1, \ldots, n)$$

is a weight homogeneous Laurent solution for this vector field. Then the leading coefficients $x_i^{(0)}$ satisfy the non-linear algebraic equations

$$\begin{aligned} \nu_1 x_1^{(0)} + f_1(x_1^{(0)}, \ldots, x_n^{(0)}) &= 0, \\ &\vdots \\ \nu_n x_n^{(0)} + f_n(x_1^{(0)}, \ldots, x_n^{(0)}) &= 0. \end{aligned}$$

Proposition 2 (continued)

On the other hand, the subsequent terms $x_i^{(k)}$ *satisfy*

$$(kI_n - \mathcal{K}(x^{(0)}))x^{(k)} = R^{(k)},$$

where

$$x^{(k)} = \begin{pmatrix} x_1^{(k)} \\ \vdots \\ x_n^{(k)} \end{pmatrix},\ R^{(k)} = \begin{pmatrix} R_1^{(k)} \\ \vdots \\ R_n^{(k)} \end{pmatrix},$$

where each $R_i^{(k)}$ *is a polynomial which depends on the variables* $x_1^{(l)}, \ldots, x_n^{(l)}$ *with* $1 \leq l \leq k-1$ *only. Also, the* (i, j)*-th entry of the* $(n \times n)$*-matrix* $\mathcal{K}$ *is the regular function on* $\mathbb{C}^n$ *defined by*

$$\mathcal{K}_{i,j} = \frac{\partial f_i}{\partial x_j} + \nu_i \delta_{i,j}.$$

The eigenvalues of $\mathcal{K}$ are called the **Kowalevski exponents**.

Example: The 2-dimensional first Painlevé equation

Let us consider the autonomous H_{I} given by the Hamiltonian

$$H_{\mathrm{I}}(q,p) = p^2 - q^3 - sq.$$

The Hamiltonian system is thus

$$\dot{q} = 2p =: f_1, \quad \dot{p} = 3q^2 + s =: f_2.$$

This is a weight-homogeneous system with the weights

$\deg(q,p)$	$\deg(H_1,s)$
$(2,3)$	$(6,4)$

We assume the following form of formal solutions

$$q(t) = \sum_{k=0}^{\infty} x_1^{(k)} t^{-2+k}, \quad p(t) = \sum_{k=0}^{\infty} x_2^{(k)} t^{-3+k}.$$

The initial terms have to satisfy the following nonlinear equations

$$2x_1^{(0)} + 2x_2^{(0)} = 0, \quad 3x_2^{(0)} + 3\left(x_1^{(0)}\right)^2 = 0.$$

These indicial equations have two solutions

$$\left(x_1^{(0)}, x_2^{(0)}\right) = (0,0) =: m_1,\ (1,-1) =: m_2.$$

The subsequent terms can be computed by solving linear equations

$$\left(kI_2 - \mathcal{K}(x^{(0)})\right)\begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix} = \begin{pmatrix} R_1^{(k)} \\ R_2^{(k)} \end{pmatrix},$$

where each $R_i^{(k)}$ is a polynomial which depends on the variables $x_1^{(l)}, x_2^{(l)}$ with $1 \le l \le k-1$. Also, matrix $\mathcal{K}$ is

$$\mathcal{K} = \begin{pmatrix} \frac{\partial f_1}{\partial q} & \frac{\partial f_1}{\partial p} \\ \frac{\partial f_2}{\partial q} & \frac{\partial f_2}{\partial p} \end{pmatrix} + \begin{pmatrix} \nu_1 & \\ & \nu_2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 6q & 0 \end{pmatrix} + \begin{pmatrix} 2 & \\ & 3 \end{pmatrix}.$$

$$\left(kI_2 - \mathcal{K}(x^{(0)})\right)\begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix} = \begin{pmatrix} R_1^{(k)} \\ R_2^{(k)} \end{pmatrix}$$

When the matrix $k\,\mathrm{Id}_n - \mathcal{K}(x^{(0)})$ is invertible, $x^{(k)}$ is uniquely determined by the preceding terms. If not, the term $x^{(k)}$ has free parameters. Therefore, the eigenvalues of $\mathcal{K}(x^{(0)})$ (the **Kowalevskaya exponents**) are important. Especially, the number of nonnegative integral Kowalevskaya exponents indicate how many parameters the series posses.
The solution starting from the initial term $m_1 = (0, 0)$ is a Taylor series

$$q(t; m_1) = \alpha + \beta t + t^2\left(3\alpha^2 + s\right) + 2\alpha\beta t^3 + t^4\left(3\alpha^3 + \frac{\beta^2}{2} + \alpha s\right) + O\left(t^5\right),$$
$$p(t; m_1) = \frac{\beta}{2} + t\left(3\alpha^2 + s\right) + 3\alpha\beta t^2 + t^3\left(6\alpha^3 + \beta^2 + 2\alpha s\right) + O\left(t^4\right).$$

Since the Kowalevski exponents are $2, 3$, the balance contains two free parameters α and β.

The level set of the momentum map is

$$H(q(t;m_1),p(t;m_2)) = -s\alpha - \alpha^3 + \frac{\beta^2}{4} = h.$$

If we write $\alpha = x$, $\beta = 2y$, the equation is

$$y^2 = x^3 + sx + h.$$

This is an elliptic curve in the Weierstrass form.

The spectral curve and the Hamiltonian for P_{I}

- The autonomous first Painlevé equation has the following Lax pair.

$$A(x) = \begin{pmatrix} -p & x^2 + qx + q^2 + s \\ x - q & p \end{pmatrix}.$$

  The spectral curve is defined by $\det(yI_2 - A(x)) = 0$, which is equivalent to

$$y^2 = x^3 + sx + H_{\mathrm{I}}.$$

- The level set $H_{\mathrm{I}}(q, p) = h$ of the Hamiltonian function itself is an elliptic curve.

$$p^2 - q^3 - sq = h,$$

  or by expressing $p = y, q = x$, we have

$$y^2 = x^3 + sx + h.$$

The degeneration of genus 1 curve

The curve we are considering has the following form.

$$y^2 = x^3 + sx + h.$$

The degeneration of this curve at $h = \infty$ can be studied in the following manner. The affine equation around $h = \infty$ is derived by transforming to $h = 1/\tilde{h}$, $y = \tilde{y}/\tilde{h}^3$, $x = \tilde{x}/\tilde{h}^2$:

$$\tilde{y}^2 = \tilde{x}^3 + s\tilde{h}^4\tilde{x} + \tilde{h}^5.$$

The discriminant and the j-invariant of the cubic are

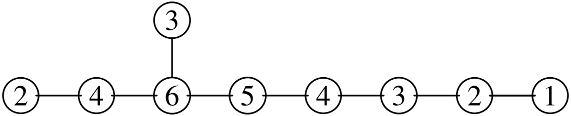
$$\begin{aligned}\Delta =& 4(s\tilde{h}^4)^3 + 27(\tilde{h}^5)^2 = \tilde{h}^{10}(4s^3\tilde{h}^2 + 27),\\ j =& \frac{4(s\tilde{h}^4)^3}{\Delta} = \frac{4s^4\tilde{h}^2}{4s^3\tilde{h}^2 + 27}.\end{aligned}$$

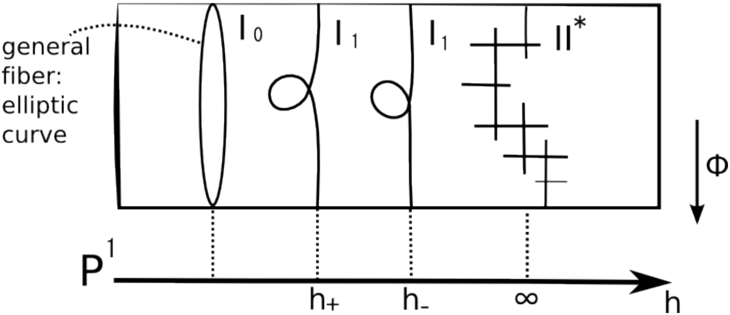
$$
\begin{aligned}
\Delta =& 4(s\tilde{h}^4)^3 + 27(\tilde{h}^5)^2 = \tilde{h}^{10}(4s^3\tilde{h}^2 + 27), \\
j =& \frac{4(s\tilde{h}^4)^3}{\Delta} = \frac{4s^4\tilde{h}^2}{4s^3\tilde{h}^2 + 27}.
\end{aligned}
$$

Kodaira	Dynkin	ord(Δ)	ord(j)	Kodaira	Dynkin	ord(Δ)	ord(j)
I_0	-	0	≥ 0	I_0^*	$D_4^{(1)}$	6	≥ 0
I_m	$A_{m-1}^{(1)}$	m	$-m$	I_m^*	$D_{4+m}^{(1)}$	$6+m$	$-m$
II	-	2	≥ 0	IV^*	$E_6^{(1)}$	8	≥ 0
III	$A_1^{(1)}$	3	≥ 0	III^*	$E_7^{(1)}$	9	≥ 0
IV	$A_2^{(1)}$	4	≥ 0	II^*	$E_8^{(1)}$	10	≥ 0

Table: Tate's algorithm and Kodaira types

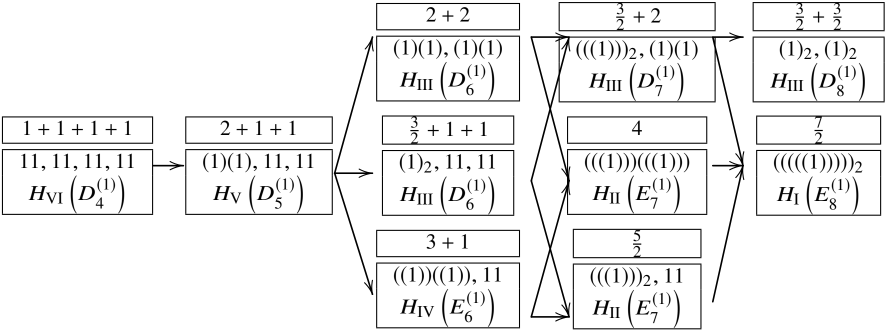
At $\tilde{h} = 0$, using Tate's algorithm, we can see that the elliptic curve has the degeneration of Kodaira-type II^* or E_8 in Dynkin's notation.



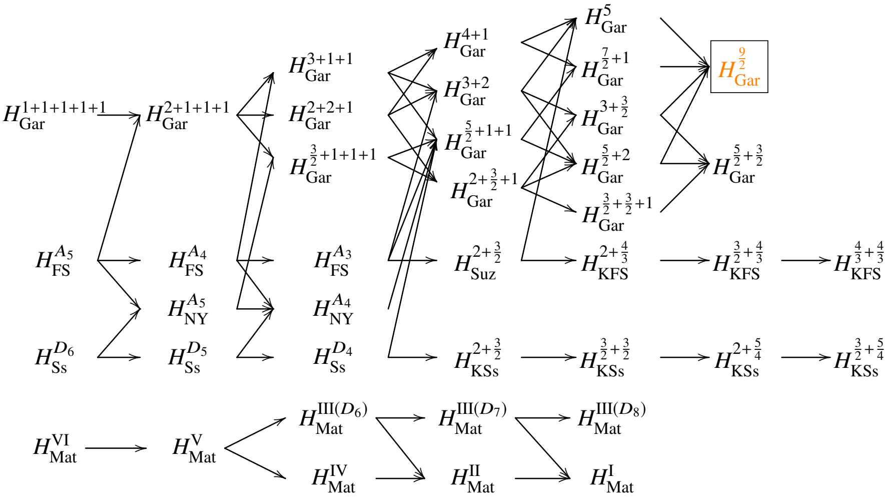


Degeneration scheme of the 2-dimensional Painlevé-type equations

There are 8 types of 2-dimensional Painlevé-type equations, and all the equations can be obtained from H_{VI} by degeneration.



Degeneartion scheme of the 4-dimensional Painlevé-type equations [10, 9, 15, 8, 5, 7, 6]



Example: The autonomous Garnier system of type 9/2 [12]

The autonomous Garnier system of type 9/2 is a Hamiltonian system with the Hamiltonians

$$
\begin{aligned}
H_1 = H^{\frac{9}{2}}_{\mathrm{Gar},s_1} &= p_1 q_2^2 - p_1 s_1 + p_2 s_2 + p_1^4 + 3 p_2 p_1^2 + p_2^2 - 2 q_1 q_2, \\
H_2 = H^{\frac{9}{2}}_{\mathrm{Gar},s_2} &= p_1^2 q_2^2 - 2 p_1 q_1 q_2 + p_2 q_2^2 + p_1^3 s_2 + p_1 s_2^2 + p_2 p_1 s_2 + p_2 s_1 - p_2 p_1^3 - 2 \\
&\quad - q_2^2 s_2 + q_1^2,
\end{aligned}
$$

where s_1, s_2 are constants. The Hamiltonian system for $H^{\frac{9}{2}}_{\mathrm{Gar},s_1}$ is

$$
\begin{aligned}
\frac{dq_1}{dt} &= 4p_1^3 + 6p_2 p_1 + q_2^2 - s_1 =: f_1, & \frac{dq_2}{dt} &= 3p_1^2 + 2p_2 + s_2 =: f_3, \\
\frac{dp_1}{dt} &= 2q_2 =: f_2, & \frac{dp_2}{dt} &= 2(q_1 - p_1 q_2) =: f_4.
\end{aligned}
$$

This is a weight-homogeneous Hamiltonian system with the following weights.

$\deg(q_1, p_1, q_2, p_2)$	$\deg(H_1, H_2, s_1, s_2)$
$(5, 2, 3, 4)$	$(8, 10, 6, 4)$

9

There are three types of family of Laurent series, with the initial terms $m_1 = (0, 0, 0, 0)$, $m_2 = (-1, 1, -1, 0)$ and $m_3 = (9, 3, -3, -9)$, respectively. The Kowalevski exponents (eigenvalues of $\mathcal{K}(x^{(0)})$, "K-exponents" for short) for each indicial locus is as follows.

indicial locus	K-exponents	# free para's	fiber (Liouville torus)
$m_1 = (0, 0, 0, 0)$	$(2, 3, 4, 5)$	4	affine abelian surface
$m_2 = (-1, 1, -1, 0)$	$(-1, 2, 5, 8)$	3	genus two curve
$m_3 = (9, 3, -3, -9)$	$(-1, -3, 8, 10)$	2	point

The following families of Laurent series starting from $m_2 = (-1, 1, -1, 0)$ contains three free parameters, α, β, and γ.

$$x_1(t; m_2) = -\frac{1}{t^5} + \frac{\alpha}{t^3} + \beta + t\left(-\frac{\alpha^3}{2} - \frac{9\alpha s_2}{35} + \frac{s_1}{7}\right) - \frac{15}{2}t^2(\alpha\beta) + \gamma t^3 + t^4\left(\frac{18\beta s_2}{7} - \frac{15\alpha^2\beta}{2}\right) + O\left(t^5\right),$$

$$x_2(t; m_2) = \frac{1}{t^2} + \frac{\alpha}{2} + t^2\left(-\frac{3\alpha^2}{4} - \frac{3s_2}{5}\right) - 4\beta t^3 + \frac{1}{28}t^4\left(-35\alpha^3 - 24\alpha s_2 + 4s_1\right) + O$$

$$x_3(t; m_2) = -\frac{1}{t^3} + t\left(-\frac{3\alpha^2}{4} - \frac{3s_2}{5}\right) - 6\beta t^2 + \frac{1}{14}t^3\left(-35\alpha^3 - 24\alpha s_2 + 4s_1\right) - \frac{15}{2}t + O\left(t^5\right),$$

$$x_4(t; m_2) = -\frac{3\alpha}{2t^2} + \left(\frac{3\alpha^2}{2} + s_2\right) + 6\beta t + t^2\left(\frac{9\alpha^3}{8} + \frac{9\alpha s_2}{10}\right) + \frac{3t^4\left(1925\alpha^4 + 1680\gamma - 120\alpha^2 s_2 - 400\alpha s_1 - 1008 s_2^2\right)}{12320} + O\left(t^5\right).$$

The level set of the moment map is

$$H_{s_1}(x_1(t;m_2),x_2(t;m_2),x_3(t;m_2),x_4(t;m_2)) = h_1,$$
$$H_{s_2}(x_1(t;m_2),x_2(t;m_2),x_3(t;m_2),x_4(t;m_2)) = h_2.$$

These are equivalent to the followings

$$\frac{405\alpha^4}{32}+\frac{81\gamma}{22}+\frac{648\alpha^2 s_2}{77}-\frac{150\alpha s_1}{77}-\frac{23s_2^2}{110}=h_1,$$
$$s_1\left(s_2-\frac{207\alpha^2}{308}\right)+\frac{81\left(35\left(99\alpha^5+48\alpha\gamma+704\beta^2\right)+760\alpha^3 s_2-1008\alpha s_2^2\right)}{24640}=h_2.$$

$$-\frac{243\alpha^5}{32}+81\beta^2+\frac{3\alpha h_1}{2}-\frac{81\alpha^3 s_2}{8}+s_1\left(\frac{9\alpha^2}{4}+s_2\right)-3\alpha s_2^2=h_2.$$

By replacing $\alpha=\frac{2}{3}x$, $\beta=\frac{1}{9}y$, the equation reads

$$y^2=x^5+3s_2x^3-s_1x^2+(2s_2^2-h_1)x+h_2-s_1s_2.$$

These three parameter family of the Laurent series corresponds to a genus two curve on a fiber of the momentum map. This curve (the boundary divisor of the Liouville torus) is called the **Painlevé divisor**.

Three types of families of Laurent solutions and restriction to a fiber

indicial locus	K-exponents	# para's	fiber (Liouville torus)	dim
m_1 = (0,0,0,0)	$(2, 3, 4, 5)$	4	affine abelian surface	4-2=2
m_2 = (-1,1,-1,0)	$(-1, 2, 5, 8)$	3	genus two curve	3-2=1
m_3 = (9,3,-3,-9)	$(-1, -3, 8, 10)$	2	point	2-2=0

$$y^2 = x^5 + 3s_2x^3 - s_1x^2 + (2s_2^2 - h_1)x + h_2 - s_1s_2.$$

The three parameter Laurent solution (principal balance) corresponds to a genus two curve (Painlevé divisor) on a fiber of the momentum map.

This equation is exactly **same as the spectral curve** of Garnier 9/2.

$$A(x) = A_0x^3 + A_1x^2 + A_2x + A_3,$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & p_1 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} q_2 & p_1^2 + p_2 + 2s_1 \\ -p_1 & -q_2 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} q_1 - p_1q_2 & p_1^3 + 2p_1p_2 - q_2^2 + s_1p_1 - s_2 \\ -p_2 + s_1 & -q_1 + p_1q_2 \end{pmatrix}.$$

The spectral curve

$$\det(yI_2 - A(x)) = 0.$$

of the Garnier system of type $\frac{9}{2}$ is expressed as

$$y^2 = x^5 + 3s_2x^3 - s_1x^2 + (2s_2^2 - h_1)x + h_2 - s_1s_2.$$

The spectral curve has the exactly the same equation as the Painlevé divisor.

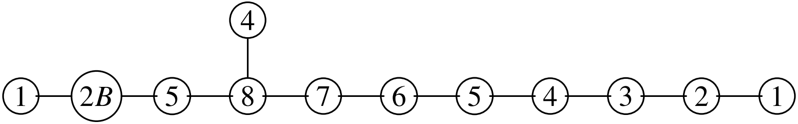
We consider the degeneration along a line $h_2 = ah_1 + b$, where a and b are generic constants.

$$y^2 = x^5 + 3s_2x^3 - s_1x^2 + (2s_2^2 - h_1)x + ah_1 + b - s_1s_2.$$

In order to see the degeneration at $h_1 = \infty$, we introduce
$\tilde{x} = x/h_1,\ \tilde{y} = y/h_1^3,\ \tilde{h} = 1/h_1.$

$$\tilde{y}^2 = \tilde{h}\left(\tilde{x}^5 + 3s_2\tilde{h}^3\tilde{x}^3 - s_1\tilde{h}^4\tilde{x}x^2 + (2s_2^2\tilde{h} - 1)\tilde{h}^4\tilde{x} + \tilde{h}^4(a + (b - s_1s_2)\tilde{h})\right).$$

The degenerations of genus two curves can be studied using Liu's algorighm [11], which is a genus two counterpart of Tate's algorithm.
$\mathrm{VII}^*: H^{\frac{9}{2}}_{\mathrm{Gar},s_1}$



Does it always work?

In the examples we have seen , the **Painlevé divisor** (boundary divisor in the compactification of the Liouville torus) and the **spectral curve** are isomorphic. Therefore, they had the same generic degeneration.

Painlevé divisor	spectral curve
traceable from nonlinear system	needs an actual linear problem (Lax)

Can we **recover the family of spectral curves from the family of the Painlevé divisors?** If so, we are able to recover the singularity data of linear problem (Lax) just be looking at the nonlinear integrable system.

Nonlinear (Painlevé divisor) $\overset{?}{\rightleftharpoons}$ **Linear** (spectral curve)

Preliminaries: Uniqueness of the polarization

The Jacobian of a smooth projective curve of genus g

$$J(C) := H^0(\omega_C)^*/H_1(C,\mathbb{Z})$$

comes with the canonical principal polarization Θ induced by the symplectic basis for C.

The classical Torelli's therem startes:

Theorem 1 (The classical Torelli theorem for curves)

Two Jacobians $(J(C),\Theta)$ *and* $(J(C'),\Theta')$ *of smooth curves* C *and* C' *are isomorphic as polarized abelian varieties if and only if* C *and* C' *are isomorphic.*

Therefore, it is enough to **show that** the typical element of our family has **unique principal polarization.** This in tern, is equivalent to saying that the Jacobian of the typical element of our family has **no nontrivial endomorphism**.

Theorem 2

For the 4-dimesional autonomous Painlevé-type equations, the Jacobian of generic spectral curve has **no nontrivial endomorphism**.

Using the correspondence of the $\mathrm{NS}(X)$ and the endomorphism ring, we have

Corolally 1

For the 4-dimesional autonomous Painlevé-type equations, the Jacobian of generic spectral curve has **unique principal polarization**.

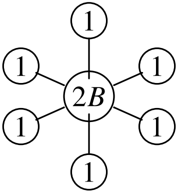
Theorem 3

For the 4-dimesional autonomous Painlevé-type equations, the generic spectral curve is isomorphic to the corresponding Painlevé divisor. In particular, **generic degeneration of the spectral curve and generic degeneration of the Painlevé divisors are the same**.

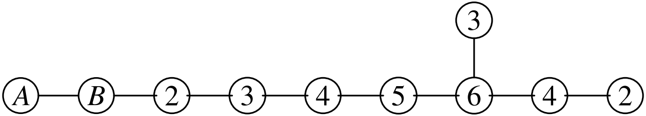
Remark 1

For the all 40 types of autonomous 4-dimensional Painlevé-type equqtions, the generic degeneration of the spectral curves are known [1]. Therefore, if we compute the generic degeneration of the Painlevé divisors for one of these equations, we can tell the corresponding linear equation.

Examples: I^*_{0-0-0} : $H^{1+1+1+1+1}_{\mathrm{Gar}}$

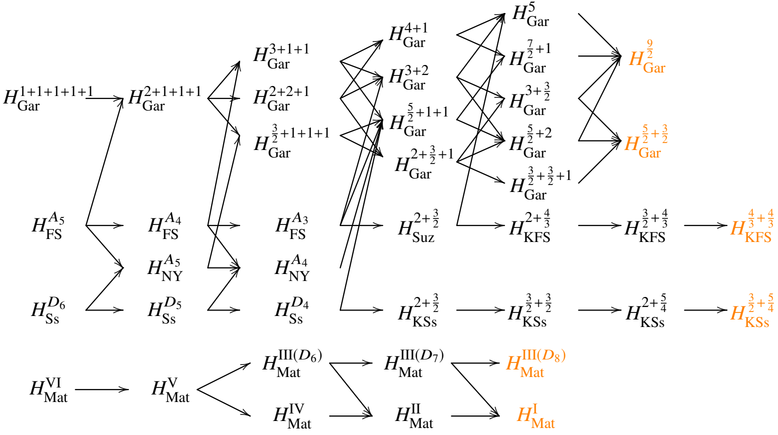


$\mathrm{I}_0 - \mathrm{II}^* - 1$: $H^{\mathrm{Mat}}_{\mathrm{I}}$



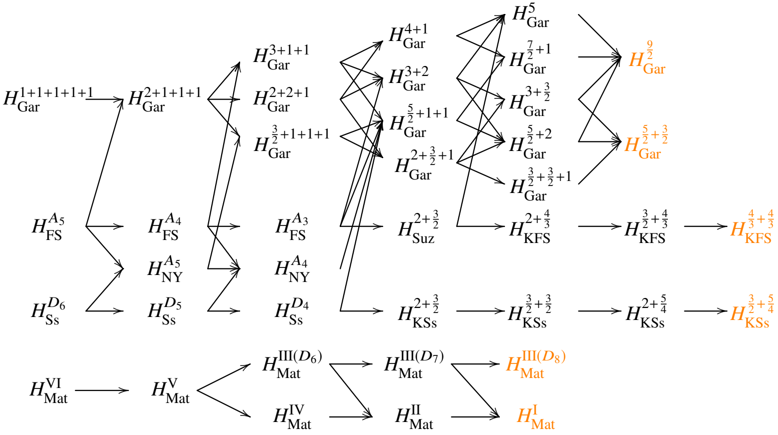
Sketch of a proof for Theorem 2

- We first prove the triviality of the endomorphism rings for the most degenerated cases ($H_{\text{Gar}}^{9/2}$, $H_{\text{Gar}}^{\frac{5}{2}+\frac{3}{2}}$, $H_{\text{KFS}}^{\frac{4}{3}+\frac{4}{3}}$, $H_{\text{KSs}}^{\frac{3}{2}+\frac{5}{4}}$, $H_{\text{Mat}}^{\text{III}(\text{D}_8)}$, $H_{\text{Mat}}^{\text{I}}$).



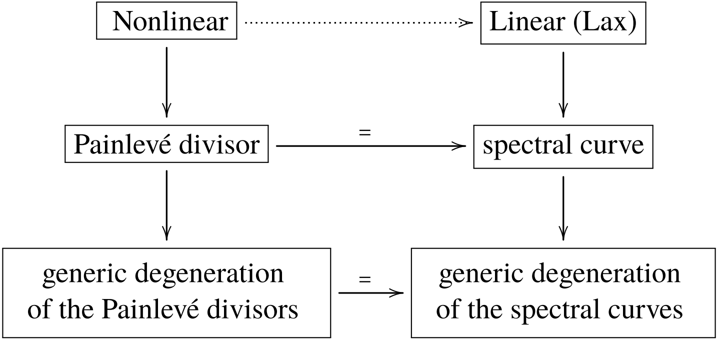
- Note that Jacobian of generic hyperelliptic curve has only trivial endomorphism.
- We will show that the space of the spectral curves of our specific system dominate the moduli space of genus two curve so that a typical curve in our family has no non-trivial endomorphisms.
- The moduli scheme of genus two curves M_2 can be identified with $\mathrm{Proj}\, k[J_2, J_4, J_6, J_{10}] - \{J_{10} = 0\}$, where J_{2i}'s are the Igusa invariants and J_{10} is the discriminant.
- For the 4 cases ($H^{9/2}_{\mathrm{Gar}}, H^{\frac{5}{2}+\frac{3}{2}}_{\mathrm{Gar}}, H^{\mathrm{III}(D_8)}_{\mathrm{Mat}}, H^{\mathrm{I}}_{\mathrm{Mat}}$), we have checked (using Jacobian criterion) that **the Igusa invariants of their spectral curves are algebraically independent**, so that the **each space of spectral curves dominates** M_2.
- Therefore, generic member of the spectral curves of these 4 cases has **only trivial endomorphism ring** for the Jacobian.

- For the rest 2 cases ($H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{4}{3}}$, $H_{\mathrm{KSs}}^{\frac{3}{2}+\frac{5}{4}}$), the above approach does not work and our current proof is a bit more subtle (use mod p reduction..., we skip).
- Since we know that all the other cases degenerate to one of these 6 cases, all the cases have generically trivial endomorphism rings.



Summary

For the autonomous 4-dimensional Painlevé-type equations, **we can recover a linear problem from a nonlinear problem** (in principle).



Thank you for your attention!

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