Representations of Yangians via Howe duality

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In honour of Vitaly Tarasov and Alexander Varchenko

References:

- Etingof - Varchenko (2002)
  Dynamical Weyl groups and applications
- Felder - Markov - Tarasov - Varchenko (2000)
  Differential equations compatible with KZ equations
- Tarasov - Varchenko (2002)
  Duality for Knizhnik-Zamolodchikov and dynamical equations

$\mathfrak{g}$ - complex semisimple Lie algebra, $\mathfrak{g} = \mathfrak{n} + \mathfrak{h} + \mathfrak{n}'$

Δ^+ - set of positive roots of $\mathfrak{g}$, $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$

$\mathfrak{S}$ - Weyl group of $\mathfrak{g}$ with shifted action on $\mathfrak{h}^*$

$$\sigma \circ \lambda = \sigma(\lambda + \rho) - \rho \quad \text{for} \quad \sigma \in \mathfrak{S} \quad \text{and} \quad \lambda \in \mathfrak{h}^*$$

$\mathrm{U}(\mathfrak{h}) \ni X$ - polynomial function on $\mathfrak{h}^*$

$X \mapsto \sigma \circ X$ - shifted action of $\sigma \in \mathfrak{S}$ on $\mathrm{U}(\mathfrak{h})$

$$(\sigma \circ X)(\lambda) = X(\sigma^{-1} \circ \lambda) \quad \text{for} \quad \lambda \in \mathfrak{h}^*$$

$\gamma : \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})/(\mathfrak{n}\,\mathrm{U}(\mathfrak{g}) + \mathrm{U}(\mathfrak{g})\,\mathfrak{n}') \cong \mathrm{U}(\mathfrak{h})$ - canonical projection

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Theorem (Harish-Chandra): $\quad \mathrm{U}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow[\gamma]{\sim} \mathrm{U}(\mathfrak{h})^{\mathfrak{S}}$

$U(\mathfrak{g}) \subset \mathrm{A}$ - associative algebra with subspace $\mathrm{V} \subset \mathrm{A}$ such that

(i) multiplication map $U(\mathfrak{g}) \otimes \mathrm{V} \to \mathrm{A} : X \otimes Y \mapsto X\,Y$ is bijective

(ii) $\mathrm{V} \subset \mathrm{A}$ is invariant and locally finite under adjoint action of $\mathfrak{g}$

$\mathrm{A} \supset \mathrm{Norm}(\mathfrak{n}\mathrm{A})$ - normalizer of the right ideal $\mathfrak{n}\,\mathrm{A} \subset \mathrm{A}$

$$Y \in \mathrm{Norm}(\mathfrak{n}\,\mathrm{A}) \quad \Longleftrightarrow \quad Y \cdot \mathfrak{n}\,\mathrm{A} \subset \mathfrak{n}\,\mathrm{A}$$

$\mathrm{R} = \mathrm{Norm}(\mathfrak{n}\,\mathrm{A}) \,/\, (\mathfrak{n}\,\mathrm{A})$ - the Mickelsson algebra of the pair $(\mathrm{A}, \mathfrak{g})$

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N - arbitrary left A-module

R acts on the space of coinvariants $N_{\mathfrak{n}} = N/(\mathfrak{n}\,N)$

$\Delta^+ \ni \alpha_1, \ldots, \alpha_r$ - simple positive roots where $r =$rank $\mathfrak{g}$

$\mathfrak{n}' \ni E_c,\ \mathfrak{n} \ni F_c,\ \mathfrak{h} \ni H_c$ for $c = 1, \ldots, r$ - Chevalley generators

$H_\alpha = \alpha^\vee \in \mathfrak{h}$ - coroot vector for any positive root $\alpha \in \Delta^+$

E_α and F_α - Cartan-Weyl basis elements of $\mathfrak{n}'$ and $\mathfrak{n}$

$$\alpha = \alpha_c \quad \Longrightarrow \quad E_\alpha = E_c, F_\alpha = F_c, H_\alpha = H_c$$

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$$P_\alpha = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(H_\alpha + \rho(H_\alpha) + 1) \ldots (H_\alpha + \rho(H_\alpha) + s)}\, F_\alpha^s\, E_\alpha^s$$

$$P = \prod_{\alpha \in \Delta^+}^{\rightarrow} P_\alpha$$ - extremal projector for $\mathfrak{g}$

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Theorem (Asherova - Smirnov - Tolstoy):

$$P^2 = P \quad \text{and} \quad E_\alpha P = P F_\alpha = 0 \quad \text{for} \quad \alpha \in \Delta^+$$

$\overline{\mathsf{U}(\mathfrak{h})} \subset \bar{\mathrm{A}}$ - rings of fractions of $\mathsf{U}(\mathfrak{h}) \subset \mathrm{A}$ with denominators set

$$\{\, H_\alpha + z \mid \alpha \in \Delta^+,\; z \in \mathbb{Z} \,\} \subset \mathsf{U}(\mathfrak{h})$$

$\mathfrak{n}\,\bar{\mathrm{A}}$ and $\bar{\mathrm{A}}\,\mathfrak{n}'$ - right and left ideals of the algebra $\bar{\mathrm{A}}$ respectively

$\bar{\mathrm{R}} = \mathrm{Norm}(\mathfrak{n}\,\bar{\mathrm{A}}) \,/\, (\mathfrak{n}\,\bar{\mathrm{A}})$ - localized Mickelsson algebra

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Proposition:

(i) $\bar{\mathrm{Z}} = \bar{\mathrm{A}}/(\mathfrak{n}\,\bar{\mathrm{A}} + \bar{\mathrm{A}}\,\mathfrak{n}')$ is a torsion-free $\overline{\mathrm{U}(\mathfrak{h})}$-bimodule, and an associative algebra with multiplication

$$A * B = A\,P\,B$$

(ii) restriction to $\bar{\mathrm{R}} \subset \bar{\mathrm{A}}/(\mathfrak{n}\,\bar{\mathrm{A}})$ of the projection $\bar{\mathrm{A}}/(\mathfrak{n}\,\bar{\mathrm{A}}) \to \bar{\mathrm{Z}}$ along $\bar{\mathrm{A}}\,\mathfrak{n}'$ is an algebra isomorphism $\bar{\mathrm{R}} \to \bar{\mathrm{Z}}$

$\mathrm{U}(\mathfrak{h}) \ni X$ - polynomial function on $\mathfrak{h}^*$

$\mathfrak{S} \ni \sigma_c$ - simple reflection corresponding to $\alpha_c \in \Delta^+$

$$\underbrace{\sigma_c\,\sigma_d\,\sigma_c\ \ldots}_{m_{cd}} = \underbrace{\sigma_d\,\sigma_c\,\sigma_d\ \ldots}_{m_{cd}} \quad \text{for} \quad c \neq d$$

$\xi_c : \mathrm{A} \to \bar{\mathrm{A}}$ - linear map defined by setting for any $Y \in \mathrm{A}$

$$\xi_c\,(Y) = \sum_{s=0}^{\infty} \frac{1}{s!\,H_c(H_c-1)\ldots(H_c-s+1)}\; E_c^s\,\mathrm{ad}_{F_c}^{\,s}(Y)$$

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Proposition:

$$\xi_c(X\,Y) \in (\sigma_c \circ X)\,\xi_c(Y) + \mathfrak{n}\,\bar{\mathrm{A}}$$

so that a linear map $\bar{\xi}_c : \bar{\mathrm{A}} \to \bar{\mathrm{A}}/(\mathfrak{n}\,\bar{\mathrm{A}})$ can be defined by setting

$$\bar{\xi}_c(X\,Y) = (\sigma_c \circ X)\,\xi_c(Y) + \mathfrak{n}\,\bar{\mathrm{A}} \quad \text{for} \quad X \in \overline{\mathrm{U}(\mathfrak{h})}$$

Proposition:

(i) $\sigma_c(\mathfrak{n}\,\bar{\mathrm{A}}) \subset \ker \bar{\xi}_c$

(ii) $\bar{\xi}_c(\,\sigma_c(\bar{\mathrm{A}}\,\mathfrak{n}')) \subset \mathfrak{n}\,\bar{\mathrm{A}} + \bar{\mathrm{A}}\,\mathfrak{n}'$

Hence the Zhelobenko operator $\check{\xi}_c :\ \bar{\mathrm{Z}} \to \bar{\mathrm{Z}}$ can be defined as $\bar{\xi}_c \cdot \sigma_c$ applied to elements of $\bar{\mathrm{A}}$ taken modulo $\mathfrak{n}\,\bar{\mathrm{A}} + \bar{\mathrm{A}}\,\mathfrak{n}'$

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Theorem (Zhelobenko):

$$\underbrace{\check{\xi}_c\,\check{\xi}_d\,\check{\xi}_c\,\dots}_{m_{cd}} = \underbrace{\check{\xi}_d\,\check{\xi}_c\,\check{\xi}_d\,\dots}_{m_{cd}} \quad \text{for} \quad c \neq d$$

Hence for any reduced decomposition $\sigma = \sigma_{c_1} \dots \sigma_{c_k}$ in $\mathfrak{S}$ the map

$$\check{\xi}_\sigma = \check{\xi}_{c_1} \dots \check{\xi}_{c_k} : \bar{\mathrm{Z}} \to \bar{\mathrm{Z}}$$

does not depend on the choice of the decomposition.

$\bar{\mathrm{Z}} \supset \bar{\mathrm{Z}}^{\mathfrak{h}}$ - invariants under adjoint action of $\mathfrak{h}$; preserved by $\check{\xi}_\sigma$

Theorem (Khoroshkin - Ogievetsky):

(i) $\check{\xi}_\sigma(A * B) = \check{\xi}_\sigma(A) * \check{\xi}_\sigma(B)$ for any $A, B \in \bar{\mathrm{Z}}$ and $\sigma \in \mathfrak{S}$

(ii) $\check{\xi}_\sigma \,|\, \bar{\mathrm{Z}}^{\mathfrak{h}}$ is an involution for $\sigma = \sigma_1, \ldots, \sigma_r$

We get an action of the Weyl group $\mathfrak{S}$ by authomorphisms of $\bar{\mathrm{Z}}^{\mathfrak{h}}$

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$\mathrm{Z}^{\,\mathfrak{h}} \subset \mathrm{Z} = \mathrm{A}\,/\,(\mathfrak{n}\,\mathrm{A} + \mathrm{A}\,\mathfrak{n}')$ - double coset vector space

$$\mathrm{Q} = \{\, A \in \mathrm{Z}^{\,\mathfrak{h}} \;|\; \check{\xi}_\sigma(A) = A \;\text{ for each }\; \sigma \in \mathfrak{S} \,\}$$

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γ maps the centralizer $\mathrm{A}^{\mathfrak{g}} \subset \mathrm{A}$ isomorphically onto Q

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Example: if $\mathrm{A} = \mathrm{U}(\mathfrak{g})$ then γ is the Harish-Chandra isomorphism

Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ - associative algebra generated by $T_{ij}^{(a)}$ where

$$i, j = 1, ..., n \quad \text{and} \quad a = 1, 2, \ldots$$

$$T_{ij}(u) = \delta_{ij} + T_{ij}^{(1)} u^{-1} + T_{ij}^{(2)} u^{-2} + \ldots \in \mathrm{Y}(\mathfrak{gl}_n)\,[[u^{-1}]]\,.$$

E_{ij} - $n \times n$ matrix units ; $\quad 1_n = E_{11} + \ldots + E_{nn}$ - identity matrix

$$T_1(u) = T(u) \otimes 1_n \quad \text{and} \quad T_2(v) = 1_n \otimes T(v)\,.$$

$$R(u) \;=\; u - \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji}$$ - Yang R-matrix

Relations in $\mathrm{Y}(\mathfrak{gl}_n)$ are written the as $n^2 \times n^2$ matrix equation

$$R(u - v)\, T_1(u)\, T_2(v) \;=\; T_2(v)\, T_1(u)\, R(u - v)\,.$$

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$\mathrm{Y}(\mathfrak{gl}_n)$ - Hopf algebra : $T_{ij}(u) \mapsto \sum_{k=1}^{n} T_{ik}(u) \otimes T_{kj}(u)$ - comultiplication

Twisted Yangian $\mathrm{Y}(\mathfrak{sp}_n)$ - subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$ generated by $S_{ij}^{(a)}$

$$S_{ij}(u) = \delta_{ij} + S_{ij}^{(1)} u^{-1} + S_{ij}^{(2)} u^{-2} + \dots$$

$$S(u) \ = \ T^t(-u)\, T(u)$$

^{t} - transposition relative to the form $\langle \, , \rangle$ on $\mathbb{C}^n$ fixed by $\mathfrak{sp}_n \subset \mathfrak{gl}_n$

$\widetilde{R}(u)$ - transpose of $R(u)$ relative to $\langle \, , \rangle$ in either tensor factor

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Relations in $\mathrm{Y}(\mathfrak{sp}_n)$ can be written as the matrix equations

$$R(u-v)\, S_1(u)\, \widetilde{R}(-u-v)\, S_2(v) = S_2(v)\, \widetilde{R}(-u-v)\, S_1(u)\, R(u-v)$$

$$S^t(u) = S(-u) - \frac{S(u) - S(-u)}{2u}$$

$\deg T_{ij}^{(a)} = a - 1$ for $a = 1, 2, \ldots$ defines ascending filtration on $\mathrm{Y}(\mathfrak{gl}_n)$

$\mathfrak{gl}_n[u] = \mathfrak{gl}_n + \mathfrak{gl}_n \cdot u + \mathfrak{gl}_n \cdot u^2 + \ldots$ - polynomial current Lie algebra

Proposition (Drinfeld):

$T_{ij}^{(a)} \mapsto E_{ij}\, u^{a-1}$ Hopf algebra isomorphism $\mathrm{gr}\,\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n[u])$

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$\mathfrak{gl}_n[u] \supset \mathfrak{t}$ - twisted polynomial current Lie algebra relative to $\langle\ ,\ \rangle$

$$\mathfrak{t} = \{\, X(u) \in \mathfrak{gl}_n\,[u] \mid X(-u) = -X^t(u)\,\}$$

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Proposition (Olshanski):

(i) $S_{ij}^{(a)} \mapsto E_{ij}\, u^{a-1} - E_{ij}^t\, (-u)^{a-1}$ isomorphism $\mathrm{gr}\, \mathrm{Y}(\mathfrak{sp}_n) \to \mathrm{U}(\mathfrak{t})$

(ii) comultiplication $\mathrm{Y}(\mathfrak{sp}_n) \to \mathrm{Y}(\mathfrak{sp}_n) \otimes \mathrm{Y}(\mathfrak{gl}_n) \neq \mathrm{Y}(\mathfrak{sp}_n)^{\otimes 2}$

$\mathcal{G}_{mn}$ - Grassmann algebra of $\mathbb{C}^{mn} = \mathbb{C}^m \otimes \mathbb{C}^n$ generated by x_{ai}

$$a = 1, \ldots, m \quad \text{and} \quad i = 1, \ldots, n$$

$$x_{ai}\, x_{bj} = -x_{bj}\, x_{ai}$$

∂_{ai} - left derivation (inner multiplication) in $\mathcal{G}_{mn}$ relative to x_{ai}

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$\mathcal{GD}_{mn}$ - associative algebra generated by left multiplications by x_{ai} and left derivations ∂_{bj} acting on $\mathcal{G}_{mn}$

$$\mathrm{U}(\mathfrak{gl}_n) \to \mathcal{GD}_{mn} : \ E_{ij} \mapsto \sum_{a=1}^{m} x_{ai}\, \partial_{aj}$$ - natural action of $\mathfrak{gl}_n$ on $\mathcal{G}_{mn}$

$$\mathrm{U}(\mathfrak{gl}_m) \to \mathcal{GD}_{mn} : \ E_{ab} \mapsto \sum_{i=1}^{n} x_{ai}\, \partial_{bi}$$ - natural action of $\mathfrak{gl}_m$ on $\mathcal{G}_{mn}$

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The images of $\mathrm{U}(\mathfrak{gl}_m)$ and $\mathrm{U}(\mathfrak{gl}_n)$ in $\mathcal{GD}_{mn}$ are mutual centralizers

Choose the form $\langle\ ,\ \rangle$ whose matrix in the standard basis of $\mathbb{C}^n$ is

$$\begin{bmatrix} 0 & & & & & 1 \\ & \ddots & & & \iddots & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & \iddots & & & \ddots & \\ -1 & & & & & 0 \end{bmatrix}$$

$\left[\delta_{\tilde{\imath} j}\,\varepsilon_i\right]_{i,j=1}^{n}$ where $\tilde{\imath} = n - i + 1$ and $\varepsilon_i = 1, -1$ for $i \leqslant n/2, i > n/2$

$\mathfrak{gl}_n \supset \mathfrak{sp}_n$ - spanned by $F_{ij} = E_{ij} - \varepsilon_i\,\varepsilon_j\,E_{\tilde{\jmath}\tilde{\imath}}$ where $i, j = 1, \ldots, n$

Choose the form $\langle\ ,\ \rangle$ whose matrix in the standard basis of $\mathbb{C}^n$ is

$$\begin{bmatrix} 0 & & & & & 1 \\ & \ddots & & & \iddots & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & \iddots & & & \ddots & \\ -1 & & & & & 0 \end{bmatrix}$$

$\left[\delta_{\tilde{\imath} j}\, \varepsilon_i\right]_{i,j=1}^{n}$ where $\tilde{\imath} = n - i + 1$ and $\varepsilon_i = 1, -1$ for $i \leqslant n/2, i > n/2$

$\mathfrak{gl}_n \supset \mathfrak{sp}_n$ - spanned by $F_{ij} = E_{ij} - \varepsilon_i\, \varepsilon_j\, E_{\tilde{\jmath}\tilde{\imath}}$ where $i, j = 1, \ldots, n$

$\mathfrak{sp}_n$ acts on $\mathcal{G}_{mn}$ by restriction from $\mathfrak{gl}_n$; for $c, d = \pm 1, \ldots, \pm m$ put

$$p_{ci} = x_{-c,i} \quad \text{and} \quad q_{ci} = \partial_{-c,i} \quad \text{if} \quad c < 0$$

$$p_{ci} = \varepsilon_i\, \partial_{c\, \tilde{\imath}} \quad \text{and} \quad q_{ci} = \varepsilon_i\, x_{c\, \tilde{\imath}} \quad \text{if} \quad c > 0$$

$\mathrm{U}(\mathfrak{sp}_n) \to \mathcal{GD}_{mn}: \; F_{ij} \mapsto -\, m\, \delta_{ij} + \sum_{c=-m}^{m} p_{ci}\, q_{cj}$ - action of $\mathfrak{sp}_n$ on $\mathcal{G}_{mn}$

Label the standard basis vectors in $\mathbb{C}^{2m}$ by $-m, \dots, -1, 1, \dots, m$

Choose symplectic form on $\mathbb{C}^{2m}$ with the matrix

$$\begin{bmatrix} 0 & & & & & 1 \\ & \ddots & & & \iddots & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & \iddots & & & \ddots & \\ -1 & & & & & 0 \end{bmatrix}$$

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Theorem (Howe):

(i) The Lie algebra $\mathfrak{sp}_{2m}$ acts on $\mathcal{G}_{mn}$ so that

$$\mathrm{U}(\mathfrak{sp}_{2m}) \to \mathcal{GD}_{mn} : \; F_{cd} \mapsto -\delta_{cd}\, n/2 + \sum_{i=1}^{n} q_{ci}\, p_{di}$$

(ii) Images of $\mathrm{U}(\mathfrak{sp}_{2m})$ and $\mathrm{U}(\mathfrak{sp}_n)$ in $\mathcal{GD}_{mn}$ - mutual centralizers

For $\mathrm{A} = \mathsf{U}(\mathfrak{gl}_m) \otimes \mathcal{GD}_{mn}$ fix diagonal embedding $\mathsf{U}(\mathfrak{gl}_m) \to \mathrm{A}$

$$E_{ab} \mapsto E_{ab} \otimes 1 + \sum_{i=1}^{n} 1 \otimes x_{ai}\, \partial_{bi}$$

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Theorem (Arakawa - Suzuki - Tsuchiya):

(i) a homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{A}^{\mathfrak{gl}_m}$ is defined by

$$T_{ij}(u) \mapsto \delta_{ij} + \sum_{a,b=1}^{m} X_{ab}(u) \otimes x_{ai}\, \partial_{bj}$$

(ii) $\mathrm{A}^{\mathfrak{gl}_m}$ is generated by $\mathrm{U}(\mathfrak{gl}_m)^{\mathfrak{gl}_m} \otimes 1$ and the image of $\mathrm{Y}(\mathfrak{gl}_n)$

$$\mathcal{F} : \ \mathfrak{gl}_m\text{-Mod} \ \to \ \mathfrak{gl}_m \times \mathrm{Y}(\mathfrak{gl}_n)\text{-Mod} : \ M \ \mapsto \ M \otimes \mathcal{G}_{mn}$$

For $\mathrm{A} = \mathrm{U}(\mathfrak{sp}_{2m}) \otimes \mathcal{GD}_{mn}$ fix diagonal embedding $\mathrm{U}(\mathfrak{sp}_{2m}) \to \mathrm{A}$

$$F_{cd} \mapsto F_{cd} \otimes 1 + 1 \otimes (-\,\delta_{cd}\, n\,/\,2 + \sum_{i=1}^{n} q_{ci}\, p_{d\,i})$$

For $\mathrm{A} = \mathrm{U}(\mathfrak{sp}_{2m}) \otimes \mathcal{GD}_{mn}$ fix diagonal embedding $\mathrm{U}(\mathfrak{sp}_{2m}) \to \mathrm{A}$

$$F_{cd} \mapsto F_{cd} \otimes 1 + 1 \otimes \Big(-\,\delta_{cd}\, n \,/\, 2 + \sum_{i=1}^{n} q_{ci}\, p_{d\,i}\Big)$$

For $F = \big[F_{cd}\big]_{c,\,d=-m}^{\,m}$ take the inverse $(u + F)^{-1} = \big[X_{cd}(u)\big]_{c,d=-m}^{\,m}$

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Theorem (Khoroshkin - Nazarov):

(i) a homomorphism $\mathrm{Y}(\mathfrak{sp}_n) \to \mathrm{A}^{\mathfrak{sp}_{2m}}$ is defined by

$$S_{ij}(u) \mapsto \delta_{ij} + \sum_{c,d=-m}^{m} X_{cd}\,(u - \tfrac{1}{2} - m) \otimes p_{ci}\, q_{dj}$$

(ii) $\mathrm{U}(\mathfrak{sp}_{2m})^{\mathfrak{sp}_{2m}} \otimes 1$ and the image of $\mathrm{Y}(\mathfrak{sp}_n)$ generate $\mathrm{A}^{\mathfrak{sp}_{2m}}$

$$\mathcal{F}:\ \mathfrak{sp}_{2m}\text{-Mod} \ \to\ \mathfrak{sp}_{2m} \times \mathrm{Y}(\mathfrak{sp}_n)\text{-Mod}:\ M \ \mapsto\ M \otimes \mathcal{G}_{mn}$$

$(\mathfrak{g}, \mathfrak{f}) = (\mathfrak{gl}_m, \mathfrak{gl}_n)$ or $(\mathfrak{sp}_{2m}, \mathfrak{sp}_n)$ - dual pair where $\mathfrak{g} = \mathfrak{n} + \mathfrak{h} + \mathfrak{n}'$

$$\mathcal{F}_\lambda : \mathfrak{g}\text{-Mod} \to \mathrm{Y}(\mathfrak{f})\text{-Mod} :$$

$$M \mapsto \mathcal{F}_\lambda(M) = \mathcal{F}(M)^{\lambda}_{\mathfrak{n}} = (M \otimes \mathcal{G}_{mn})^{\lambda}_{\mathfrak{n}} \quad \text{for} \quad \lambda \in \mathfrak{h}^*$$

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Example: for $(\mathfrak{g},\mathfrak{f}) = (\mathfrak{gl}_m,\mathfrak{gl}_n)$ and $M = M_\mu$ - Verma module,

the $\mathrm{Y}(\mathfrak{f})$-module $\mathcal{F}_\lambda(M_\mu)$ is equivalent to the tensor product

$$\Lambda^{\lambda_1-\mu_1}_{\mu_1} \otimes \Lambda^{\lambda_2-\mu_2}_{\mu_2-1} \otimes \ldots \otimes \Lambda^{\lambda_m-\mu_m}_{\mu_m-m+1}$$

$(\lambda_1,\ldots,\lambda_m)$ and $(\mu_1,\ldots,\mu_m)$ - labels of the weights $\lambda,\mu \in \mathfrak{h}^*$;

$\Lambda^d_z \ = \ d$-th exterior power of $\mathbb{C}^n \ = \ $ subspace in $\mathcal{G}_n$ of degree d

$\mathrm{Y}(\mathfrak{gl}_n)$-action defined by $T_{ij}(u) \mapsto \delta_{ij} + x_i\,\partial_j/(u+z)$ for $z \in \mathbb{C}$;

assuming that $\Lambda^d_z = \{0\}$ if $d \neq 0,1,2,\ldots$

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The algebra $\bar{\mathrm{Z}}$ acts on $\mathcal{F}(M_\mu)_{\mathfrak{n}}$ via the isomorphism $\bar{\mathrm{Z}} \to \bar{\mathrm{R}}$

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Proposition (Tarasov - Varchenko, Khoroshkin - Nazarov):

for generic λ the automorphism $\check{\xi}_0$ determines an intertwiner

$$\mathcal{F}_\lambda(M_\mu) \to \mathcal{F}_\lambda(M_\mu)^*$$

of Y($\mathfrak{f}$)-modules, where $\mathcal{F}_\lambda(M_\mu)^*$ is the dual module to $\mathcal{F}_\lambda(M_\mu)$

Example: for $(\mathfrak{g},\mathfrak{f}) = (\mathfrak{gl}_m,\mathfrak{gl}_n)$ and any λ,μ the Y($\mathfrak{f}$)-module

$$\mathcal{F}_\lambda(M_\mu)^* \cong \Lambda^{\lambda_m-\mu_m}_{\mu_m-m+1} \otimes \ldots \otimes \Lambda^{\lambda_2-\mu_2}_{\mu_2-1} \otimes \Lambda^{\lambda_1-\mu_1}_{\mu_1} \cong \mathcal{F}_{\sigma_0\circ\lambda}(M_{\sigma_0\circ\mu})$$

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Theorem (Khoroshkin-Nazarov):

(i) the automorphism $\check{\xi}_0$ of $\bar{\mathrm{Z}}$ determines Y$(\mathfrak{f})$-intertwiner

$$\mathcal{F}_\lambda(M_\mu) \to \mathcal{F}_\lambda(M_\mu)^*$$

(ii) the image of this intertwiner is non-zero and Y$(\mathfrak{f})$-irreducible

(iii) up to an action of the centre of Y$(\mathfrak{f})$, every irreducible finite-dimensional Y$(\mathfrak{f})$-module arises from (ii) for some λ, μ

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- (i,ii) also extend to the dual pairs $(\mathfrak{gl}_m, \mathfrak{gl}_n)$ and $(\mathfrak{sp}_{2m}, O_n)$, $(\mathfrak{so}_{2m}, \mathfrak{sp}_n)$ on the space of polynomials in *mn* commuting variables; the last two dual pairs arise (Howe) from the Weil representation of the real symplectic group Sp_{2mn}