Abelianisation of SL(2)-Connections and the Hitchin System

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15 August 2019

Abelianisation of Logarithmic SL(2)-Connections

LET: $(\mathsf{X}, \mathsf{D}) :=$ (compact) Riemann surface with reduced divisor

FIX: $a_{\mathsf{D}} :=$ generic residue data in $\mathfrak{sl}_2$ along D

DEFINE: $\mathsf{Conn}_{\mathsf{X}} := \left\{ (\mathcal{E}, \nabla) \;\middle|\; \begin{array}{c} \text{log-}\mathfrak{sl}_2\text{-connections on } (\mathsf{X}, \mathsf{D}) \\ \text{with residues } a_{\mathsf{D}} \end{array} \right\}$

CHOOSE: spectral data $a =$ (quadratic differential on X) lifting a_{D}

GET: (1) spectral curve $\Sigma \xrightarrow{\pi} \mathsf{X}$ with canonical differential η

(2) Stokes graph Γ on X ($\Leftrightarrow$ triangulation of X)

$\Rightarrow$: subcategory $\mathsf{Conn}_{\mathsf{X}}(\Gamma) := \left\{ (\mathcal{E}, \nabla) \;\middle|\; \Gamma\text{-transversality} \right\} \subset \mathsf{Conn}_{\mathsf{X}}$

Theorem (N 2019 | arXiv:1902.03384)

There is a natural equivalence of categories, called ***abelianisation***

$$\mathsf{Conn}_{\mathsf{X}}(\Gamma) \underset{\pi^{\Gamma}_{\mathrm{ab}}}{\overset{\pi^{\mathrm{ab}}_{\Gamma}}{\rightleftarrows}} \mathsf{Conn}_{\Sigma} := \left\{ \begin{array}{c} \textit{abelian log-connections} \\ (\mathcal{L}, \partial) \textit{ on } (\Sigma, \pi^* \mathsf{D} \cup \mathsf{R}) \\ \textit{with residues } \pi^* a_{\mathsf{D}} \textit{ on } \pi^* \mathsf{D} \end{array} \right\}$$

BASIC IDEA: $\mathcal{L}$ glued on Σ from pieces of Levelt filtration data

Voros Cocycle

Theorem (N 2019 | arXiv:1902.03384)

*There is a natural equivalence of categories, called **abelianisation***

$$\mathrm{Conn}_{\mathsf{X}}(\Gamma) \underset{\pi^{\Gamma}_{\mathrm{ab}}}{\overset{\pi^{\mathrm{ab}}_{\Gamma}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathrm{Conn}_{\Sigma} \qquad (\mathcal{E}, \nabla) \longmapsto (\mathcal{L}, \partial)$$

<u>KEY</u>: encode Stokes graph Γ in cohomology as ***Voros cocycle***:

$$\exists! \quad \mathbb{V} \in \check{\mathsf{Z}}^1\big(\mathsf{X}, \mathcal{A}ut(\pi_*)\big)$$

such that $\pi^{\Gamma}_{\mathrm{ab}} = \mathbb{V} \cdot \pi_*$ is inverse equivalence to $\pi^{\mathrm{ab}}_{\Gamma}$

- $\mathbb{V} = \left\{ \begin{bmatrix} 1 & \mathrm{Hol}_\gamma(-) \\ 0 & 1 \end{bmatrix} \;\middle|\; \gamma = \begin{matrix}\text{paths on } \Sigma \\ \text{canonically determined by} \\ \text{the Stokes graph } \Gamma\end{matrix} = \begin{matrix}\text{“half-cycles”} \\ \text{of } \Sigma\end{matrix} \right\}$
- $\mathbb{V}_\gamma(\mathcal{L}, \partial) = \begin{bmatrix} 1 & \mathrm{Hol}_\gamma(\partial) \\ 0 & 1 \end{bmatrix}$
- abelianisation of $\mathbb{V}$: $\exists! \; \mathbb{A} := \mathrm{Hol}(-) \in \check{\mathsf{Z}}^1(\Sigma, \mathcal{H})$ s.t. $\mathbb{V} = \mathbb{1} + \pi_* \mathbb{A}$.
- $\mathcal{H} := \mathcal{H}om(\mathrm{id}, \sigma^*)$ where $\sigma =$ canonical involution $\Sigma \to \Sigma$.

quantum curve := family $(\mathcal{E}_\hbar, \nabla_\hbar)$ for $\hbar \in (\text{sector}) \subset \mathbb{C}_\hbar$ s.t.

$$\nabla_\hbar(fe) = f\nabla_\hbar e + \hbar\,\mathrm{d}f \otimes e$$

<u>KEY PROPERTY</u>: $\lim_{\hbar \to 0} (\mathcal{E}_\hbar, \nabla_\hbar) = (E, \phi)$ Higgs bundle

Theorem (N 2019 | (in preparation))

$$\mathrm{Conn}_X^\hbar(\Gamma) \underset{\pi^\Gamma_{\mathrm{ab}}}{\overset{\pi^{\mathrm{ab}}_\Gamma}{\rightleftarrows}} \mathrm{Conn}_\Sigma^\hbar \quad (\sim)$$

$$\begin{tikzcd}
(\mathcal{E}_\hbar, \nabla_\hbar) \arrow[r, "\pi^{\mathrm{ab}}_\Gamma"] \arrow[d, "\lim_{\hbar\to 0}"'] \arrow[dr, phantom, "\circlearrowright"] & (\mathcal{L}_\hbar, \partial_\hbar) \arrow[d, "\lim_{\hbar\to 0}"] \\
(E, \phi) \arrow[r, "\textit{spectral}", "\textit{correspondence}"'] & (L, \eta)
\end{tikzcd}$$

$$\begin{pmatrix} \text{Voros data } \mathbb{V} \\ \text{or } (\pi^{\mathrm{ab}}_\Gamma, \pi^\Gamma_{\mathrm{ab}}) \end{pmatrix} = \left(\begin{matrix} \text{cocycle of } \hbar\text{-holonomies on } \Sigma \\ \Delta_\gamma = \mathrm{Par}_\gamma(-) \underset{\hbar \to 0}{\sim} \exp\left(\displaystyle\int_\gamma \eta/\hbar + \cdots \right) \end{matrix} \right)$$

Abelianisation, Stokes Data, Hitchin System

<u>!!!</u> <u>very speculative slide!</u>

$*$ related to work in progress with Anton Alekseev,
and to work in progress with with Marta Mazzocco.

- $\begin{pmatrix}\text{Stokes}\\\text{data}\end{pmatrix} = \begin{pmatrix}\text{Voros}\\\text{data}\end{pmatrix}\Bigg|_{\substack{\text{singular}\\\text{locus}}}$
- $\begin{pmatrix}\text{Voros}\\\text{data}\end{pmatrix} = \begin{pmatrix}\text{Stokes}\\\text{data}\end{pmatrix} + \begin{pmatrix}\text{connection matrix}\\\text{data}\end{pmatrix}$
- $\begin{pmatrix}\hbar\text{-leading order}\\\text{of Voros data}\end{pmatrix} = \begin{pmatrix}\text{periods of}\\\text{spectral curve}\end{pmatrix} = \begin{pmatrix}\text{action variables}\\\text{on a}\\\text{Hitchin system}\end{pmatrix}$
- $\begin{pmatrix}\hbar\text{-}\textbf{sub}\text{leading order}\\\text{of Voros data}\end{pmatrix} = \begin{pmatrix}\text{angle variables}\\\text{on a}\\\text{Hitchin system}\end{pmatrix} \underset{??}{+} \begin{pmatrix}\text{geometric}\\\text{corrections}\end{pmatrix}$

Thank you for your attention!