On fusion rules and intertwining operators for the Weyl vertex algebra

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Znanstveni centar izvrsnosti
za kvantne i kompleksne sustave te
reprezentacije Liejevih algebri

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Introduction

- **fusion rule** = dimension of the vector space of intertwining operators between three irreducible modules
- **determine fusion rules** = determine the exact decomposition of the tensor product of two modules of vertex algebra into a direct sum of irreducible representations
- **goal**: describe fusion rules in the category of weight modules for the Weyl vertex algebra (confirm Verlinde conjecture by Ridout-Wood) and relate to results for $\widehat{\mathfrak{gl}(1|1)}$

- Let V be a conformal vertex algebra with the conformal vector ω and let $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$. We assume that the derivation in the vertex algebra V is $D = L(-1)$. A **weak V-module** is a vector space M endowed with a linear map Y_M from V to the space of $End(M)$-valued fields

$$a \mapsto Y_M(a, z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}$$

such that:

1. $Y_M(|0\rangle, z) = I_M$,
2. for $a, b \in V$,

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(a, z_1) Y_M(b, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(b, z_2) Y_M(a, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(a, z_0)b, z_2).$$

Introduction

- Given three V-modules M_1, M_2, M_3, an **intertwining operator of type** $\binom{M_3}{M_1\ M_2}$ is a map $I : a \mapsto I(a,z) = \sum_{n\in\mathbb{Z}} a^I_{(n)} z^{-n-1}$ from M_1 to the space of $Hom(M_2, M_3)$-valued fields such that:
  1. for $a \in V$, $b \in M_1$, $c \in M_2$, the following Jacobi identity holds:
  $$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_{M_3}(a, z_1) I(b, z_2)c - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) I(b, z_2) Y_{M_2}(a, z_1)c$$
  $$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) I(Y_{M_1}(a, z_0)b, z_2)c,$$
  2. for every $a \in M_1$,
  $$I(L(-1)a, z) = \frac{d}{dz} I(a, z).$$
- $I\binom{M_3}{M_1\ M_2}$ = the space of intertwining operators of type $\binom{M_3}{M_1\ M_2}$
- $N^{M_3}_{M_1,M_2} = \dim I \begin{pmatrix} M_3 \\ M_1\ \ M_2 \end{pmatrix}$ = **fusion coefficient** (when finite).

Introduction

- Let M_1, M_2 be irreducible V-modules in the category K of $L(0)$-diagonalizable modules. Given n irreducible V-modules W_i, $i = 1, \ldots, n$ in K, we will say that the **fusion rule**

$$M_1 \times M_2 = \sum_{i=1}^{n} W_i$$

holds in K if $N_{M_1,M_2}^{W_i} = 1$, $i = 1, \ldots, n$, and $N_{M_1,M_2}^{R} = 0$ for any other irreducible V-module R in K which is not isomorphic to W_i, $i = 1, \ldots, n$.

Proposition
Let g be an automorphism of the vertex algebra V satisfying the conditions

$$\omega - g(\omega) \in Im(D), \quad \omega - g^{-1}(\omega) \in ImD. \tag{1}$$

Let M_1, M_2, M_3 be V–modules and $I(\cdot, z)$ an intertwining operator of type $\binom{M_3}{M_1\ M_2}$. Then we have an intertwining operator I^g of type $\binom{M_3^g}{M_1^g\ M_2^g}$, such that $I^g(b, z_1) = I(b, z_1)$, for all $b \in M_1$ and $N_{M_1,M_2}^{M_3} = N_{M_1^g,M_2^g}^{M_3^g}$.

Weyl vertex algebra (= $\beta\gamma$ vertex algebra)

- The **Weyl algebra** $\widehat{\mathcal{A}}$ is an associative algebra with generators

$$a(n), a^*(n) \quad (n \in \mathbb{Z})$$

and relations ($n, m \in \mathbb{Z}$)

$$[a(n), a^*(m)] = \delta_{n+m,0}, \quad [a(n), a(m)] = [a^*(m), a^*(n)] = 0.$$

- M = simple $\widehat{\mathcal{A}}$**–module** generated by the cyclic vector $\mathbf{1}$ s. t.

$$a(n)\mathbf{1} = a^*(n+1)\mathbf{1} = 0 \quad (n \geq 0).$$

- There is a unique vertex algebra $(M, Y, \mathbf{1})$ generated by the fields $Y(a(-1)\mathbf{1}, z) = a(z)$ and $Y(a^*(0)\mathbf{1}, z) = a^*(z)$.
- Vertex algebra M admits a family of Virasoro vectors

$$\omega_\mu = (1-\mu)a(-1)a^*(-1)\mathbf{1} - \mu a(-2)a^*(0)\mathbf{1} \quad (\mu \in \mathbb{C}),$$

so we have given a conformal vertex algebra structure to it.

Weyl vertex algebra

- A module W for the Weyl vertex algebra M is called **weight** if the operators $\beta(0)$ and $L(0)$ act semisimply on W.
- For every $s \in \mathbb{Z}$ the Weyl algebra $\widehat{\mathcal{A}}$ admits the following automorphism of $\widehat{\mathcal{A}}$

  $$\rho_s(a(n)) = a(n+s), \quad \rho_s(a^*(n)) = a^*(n-s).$$

  which can be lifted to an automorphism of the vertex algebra M and we call it **spectral flow automorphism**.
- The first Weyl algebra A_1 is generated by x, ∂_x with the commutation relation $[\partial_x, x] = 1$.
- For every $\lambda \in \mathbb{C}$,

  $$U(\lambda) = x^\lambda \mathbb{C}[x, x^{-1}]$$

  has the structure of an A_1–module..
- Let $\widehat{\mathcal{A}}_{\geq 0} = \mathbb{C}[a(n), a^*(m) \mid n, m \in \mathbb{Z}_{\geq 0}]$ be a subalgebra of $\widehat{\mathcal{A}}$. Then $U(\lambda)$ is an $\widehat{\mathcal{A}}_{\geq 0}$–module and we have the induced module for $\widehat{\mathcal{A}}$:

  $$\widetilde{U(\lambda)} = \widehat{\mathcal{A}} \otimes_{\widehat{\mathcal{A}}_{\geq 0}} U(\lambda)$$

Proposition (A)
For every $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, $\widetilde{U(\lambda)}$ is an irreducible weight module for the Weyl vertex algebra M.

Let $\mathcal{K}$ be the category of weight M–modules such that the operators $\beta(n)$, $n \geq 1$, act locally nilpotent on each module N in $\mathcal{K}$.

Theorem
Assume that $\lambda, \mu, \lambda + \mu \in \mathbb{C} \setminus \mathbb{Z}$. Then we have:

(i) $\rho_{\ell_1}(M) \times \rho_{\ell_2}(M) = \rho_{\ell_1+\ell_2}(M)$,

(ii) $\rho_{\ell_1}(M) \times \rho_{\ell_2}(\widetilde{U(\lambda)}) = \rho_{\ell_1+\ell_2}(\widetilde{U(\lambda)})$,

(iii) $\rho_{\ell_1}(\widetilde{U(\lambda)}) \times \rho_{\ell_2}(\widetilde{U(\mu)}) = \rho_{\ell_1+\ell_2}(\widetilde{U(\lambda + \mu)}) + \rho_{\ell_1+\ell_2-1}(\widetilde{U(\lambda + \mu)})$.

Sketch of proof

1. Lattice $L = \mathbb{Z}\alpha + \mathbb{Z}\beta,\ \langle \alpha, \alpha\rangle = -\langle \beta, \beta\rangle = 1,\ \langle \alpha, \beta\rangle = 0$, associated LVSA $V_L = M_{\alpha,\beta}(1) \otimes \mathbb{C}[L]$, $\Pi(0) = M_{\alpha,\beta}(1) \otimes \mathbb{C}[\mathbb{Z}(\alpha+\beta)] \subset V_L$, $M \subset \Pi(0)$.
2. Intertwining operator of type $\begin{pmatrix} \Pi_{r_1+r_2}(\lambda+\mu) \\ \Pi_{r_1}(\lambda)\ \Pi_{r_2}(\mu) \end{pmatrix}$ for the vertex algebra $\Pi(0)$ (Dong-Lepowsky), consider its restriction to M, obtain another IO by using first Proposition.
3. Affine Lie superalgebra $\mathfrak{gl}(1|1)$, associated vertex algebra $V_1(\mathfrak{gl}(1|1))$ whose fusion rules are known (Creutzig-Ridout), F Clifford algebra (bc-system) and $\mathcal{U} = M \otimes F$. We have (Kac): $V_1(\mathfrak{g}) \cong \mathcal{U}^0 = \mathrm{Ker}_{M\otimes F} E(0)$.

Sketch of proof

4.

Theorem
Assume that $r \in \mathbb{Z}$, $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. Then we have:

(i) $S\Pi_r(\lambda)$ *is an irreducible* $M \otimes F$*–module,*

(ii) $S\Pi_r(\lambda)$ *is a completely reducible* $\widehat{gl(1|1)}$*–module:*

$$S\Pi_r(\lambda) \cong \bigoplus_{s\in\mathbb{Z}} U(\hat{\mathfrak{g}}).e^{r(\beta+\gamma)+(\lambda+s)(\alpha+\beta)} \cong \bigoplus_{s\in\mathbb{Z}} \widehat{\mathcal{V}}_{r+\frac{1}{2}(\lambda+s),-\lambda-s}.$$

5.

Theorem
Assume that $\lambda_1, \lambda_2, \lambda_1+\lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$, $r_1, r_2, r_3 \in \mathbb{Z}$. Then
$\dim I\binom{S\Pi_{r_3}(\lambda_3)}{S\Pi_{r_1}(\lambda_1) \quad S\Pi_{r_2}(\lambda_2)} \le 1.$

Assume that there is a non-trivial intertwining operator of type
$\binom{S\Pi_{r_3}(\lambda_3)}{S\Pi_{r_1}(\lambda_1) \quad S\Pi_{r_2}(\lambda_2)}$ *in the category of $M \otimes F$–modules. Then*
$\lambda_3 = \lambda_1 + \lambda_2$ *and* $r_3 = r_1 + r_2$, *or* $r_3 = r_1 + r_2 - 1$.

Future work

In our future work we would like to study the following:

- Consider generalized weight modules such that their weight spaces are all ∞-dimensional,
- Include Whittaker modules into the fusion category
- Extend work to $\mathfrak{gl}(n|m)$.

Thank you!