Motivic Amplitudes

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Scalar Feynman Graphs

Let G be a **scalar Feynman graph** with n_G internal edges, l_G independent loops, $\{p_j\}$ momenta of external legs, $\{m_e\}$ masses of internal edges.

In 4D spacetime, the **parametric Feynman integral** of G is equivalent to the projective integral

$$I_G(\{p_j, m_e\}) = \int_\sigma \frac{\Omega}{\Psi_G^2} \left(\frac{\Psi_G}{\Xi_G(\{p_j, m_e\})} \right)^{n_G - 2l_G}$$

where $\sigma = \{[x_1 : \ldots : x_{n_G}] \in \mathbb{P}^{n_G - 1}(\mathbb{R}) \mid x_e \geq 0,\ e = 1, \ldots, n_G\}$

and $$\Omega = \sum_{e=1}^{n_G} (-1)^e\, x_e\, dx_1 \wedge \ldots \wedge \widehat{dx_e} \wedge \ldots \wedge dx_{n_G}\,.$$

Primitive Log-Divergent Graphs

G is called **logarithmically divergent** if it satisfies $n_G = 2l_G$.

The Feynman integral simplifies to $I_G = \int_\sigma \frac{\Omega}{\Psi_G^2}$.

Theorem. Let G be a logarithmically divergent graph. The integral I_G converges if and only if every proper subgraph $\varnothing \neq \gamma \subsetneq G$ satisfies $n_\gamma > 2l_\gamma$.

If every $\varnothing \neq \gamma \subsetneq G$ satisfies $n_\gamma > 2l_\gamma$, G is called **primitive log-divergent**.

Particular attention is given to primitive log-divergent graphs in **scalar ϕ^4 quantum field theory**.

Numeric Periods

Definition (Kontsevich, Zagier). A **numeric period** is a complex number whose real and imaginary parts are values of absolutely convergent integrals of the form

$$\int_\sigma f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

where f is a rational function with rational coefficients and $\sigma \subseteq \mathbb{R}^n$ is defined by finite unions and intersections of domains of the form $\{g(x_1, \ldots, x_n) \geq 0\}$ with g a rational function with rational coefficients.

$$\bar{\mathbb{Q}} \subset \mathscr{P} \subset \mathbb{C}$$

Examples.

- Algebraic numbers, logarithms of algebraic numbers, π
- Elliptic integrals, multiple zeta values
- Special values of hypergeometric functions and modular forms
- Values of various kinds of L-functions
- Feynman integrals

Numeric Periods

Definition. Let X be a smooth quasi-projective variety over $\mathbb{Q}$, $Y \subset X$ a subvariety, ω a close algebraic differential n-form on X vanishing on Y, and γ a singular n-chain on the complex manifold $X(\mathbb{C})$ with boundary in $Y(\mathbb{C})$. The integral $\int_\gamma \omega \in \mathbb{C}$ is a numeric period.

A period can be associated to **different integral representations**. The first step towards a unique algebraic identification is

$$\omega \quad \longmapsto \quad [\omega] \in H^n_{alg-dR}(X, Y)$$

$$\gamma \quad \longmapsto \quad [\gamma] \in H^B_n(X, Y)$$

Hodge Structures

Theorem (Grothendieck). Let X be a smooth affine variety over $\mathbb{Q}$. Then the map

$$\mathrm{comp}: \quad H^n_{alg-dR}(X) \otimes_{\mathbb{Q}} \mathbb{C} \quad \longrightarrow \quad H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

is an isomorphism, called **comparison isomorphism**.

The comparison isomorphism is induced by the **pairing**

$$H^n_{alg-dR}(X, \mathbb{Q}) \otimes H_n^{sing}(X(\mathbb{C}), \mathbb{Q}) \longrightarrow \mathbb{C}$$

$$[\omega] \otimes [\gamma] \longmapsto \int_\gamma \omega$$

The **Hodge structure** $H^n(X) = (H^n_{alg-dR}(X), H^n_B(X), \mathrm{comp})$ selects the content shared by the different cohomologies of X.

Motivic Periods

Analogously, the **motivic representation** of a period singles out its cohomological content

$$\int_\gamma \omega \quad \longmapsto \quad [H^n(X,Y),[\omega],[\gamma]]^{\mathrm{m}}$$

After factorisation modulo bilinearity, change of variables and Stokes formula, the set of motivic representations of periods identifies the **algebra of motivic periods** $\mathscr{P}^{\mathrm{m}}$.

The evaluation homomorphism $\mathscr{P}^{\mathrm{m}} \to \mathscr{P}$, called **period map**, is an isomorphism only conjecturally.

Examples

Example of $2\pi i$ **:**

$$(2\pi i)^{\mathrm{m}} = \left[H^1(\mathbb{G}_m), \left[\frac{dx}{x}\right], [\gamma_0] \right]^{\mathrm{m}} \longmapsto \quad 2\pi i = \oint_{\gamma_0} \frac{dx}{x}$$

where γ_0 is any counterclockwise cycle encircling the origin in $\mathbb{C}$.

Example of $\log(z)$, $z \in \bar{\mathbb{Q}} \setminus \{1\}$ **:**

$$\log(z)^{\mathrm{m}} = \left[H^1(\mathbb{G}_m, \{1, z\}), \left[\frac{dx}{x}\right], [\gamma_1] \right]^{\mathrm{m}} \longmapsto \quad \log(z) = \int_1^z \frac{dx}{x}$$

where γ_1 is the directed segment from 1 to z.

Motivic Feynman Integrals

$$\mathscr{P}_{log} = \mathbb{Q}\langle\ I_G \mid G \text{ is primitive log-divergent } \rangle$$

$$\mathscr{P}_{\phi^4} = \mathbb{Q}\langle\ I_G \mid G \text{ is primitive log-divergent in } \phi^4\ \rangle$$

$$\mathscr{P}_{\phi^4} \subseteq \mathscr{P}_{log} \subseteq \mathscr{P}$$

Promoting $I_G \in \mathscr{P}_{log}$ to its motivic version $I_G^{\mathfrak{m}} \in \mathscr{P}_{log}^{\mathfrak{m}}$, the presence of singularities requires special treatment via the **blow up** technique.

When applicable, it produces a well-defined motivic representation

$$I_G^{\mathfrak{m}} = [\, H^{n_G - 1}(P^G \setminus Y_G),\, D \setminus (D \cap Y_G),\, [\hat{\omega}],\, [\hat{\sigma}]\,]^{\mathfrak{m}}$$

Tannakian Formalism

Definition. A **Tannakian category** over the field $\mathbb{K}$ is a rigid abelian $\mathbb{K}$-linear tensor category $\mathcal{T}$ such that $\mathrm{End}(1) = \mathbb{K}$ and there exists an exact faithful $\mathbb{K}$-linear tensor functor $\omega : \mathcal{T} \to \mathrm{Vec}_{\mathbb{K}}$, called **fibre functor**.

Let R be a $\mathbb{K}$-algebra.
Denote $\omega_1, \omega_2 : \mathcal{T} \to \mathrm{Vec}_{\mathbb{K}}$ two fibre functors of $\mathcal{T}$.

$$\underline{\mathrm{Isom}}^{\otimes}(\omega_1, \omega_2)(R) = \left\{ \begin{array}{r} \lambda_M : \omega_1(M) \otimes_{\mathbb{K}} R \to \omega_2(M) \otimes_{\mathbb{K}} R, \\ \forall M \in \mathrm{Ob}(\mathcal{T}),\ \text{such that } \lambda_M \\ \text{is an isomorphism compatible} \\ \text{with } \otimes-\text{product and functorial} \end{array} \right\}$$

The functor $R \mapsto \underline{\mathrm{Isom}}^{\otimes}(\omega_1, \omega_2)(R)$, denoted $\underline{\mathrm{Isom}}^{\otimes}(\omega_1, \omega_2)$, is representable by an affine scheme over $\mathbb{K}$.

Tannakian Formalism

When $\omega_1 = \omega_2 = \omega$, $\underline{\mathrm{Isom}}^{\otimes}(\omega, \omega)(R)$ is written as $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$.

The functor $R \mapsto \underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ is representable by an affine group scheme over $\mathbb{K}$ and is denoted $\underline{\mathrm{Aut}}^{\otimes}(\omega) = G^{\omega}$. This is the **Tannaka group** of the pair $(\mathcal{T}, \omega)$.

Theorem. Let $\mathcal{T}$ be a Tannakian category over $\mathbb{K}$ and let ω be one of its fibre functors. The functor $\mathcal{T} \longrightarrow \mathrm{Rep}_{\mathbb{K}}(G^{\omega})$ sending X to the vector space $\omega(X)$ with the natural action of G^{ω} on $\omega(X)$, $\forall X \in \mathrm{Ob}(\mathcal{T})$, is an equivalence of categories.

Tannakian categories are indeed the categories of finite-dimensional linear representations of a pro-algebraic group.

Category of Motives

The category of Hodge structures $\mathscr{M}$ is a Tannakian category over $\mathbb{Q}$ with fibre functors $\omega_{dR}, \omega_B : \mathscr{M} \to \mathrm{Vec}_{\mathbb{Q}}$.

Denote $G^{dR} = \underline{\mathrm{Aut}}^{\otimes}(\omega_{dR})$. This is called **motivic Galois group**.

The category of motives is isomorphic to the category of finite-dimensional $\mathbb{Q}$-linear representations of the motivic Galois group

$$\mathscr{M} \simeq \mathrm{Rep}_{\mathbb{Q}}(G^{dR})$$

The same category is built from the motivic Galois group in the Betti realisation $G^B = \underline{\mathrm{Aut}}^{\otimes}(\omega_B)$.

Category of Motives

The space of motivic periods is re-expressed as

$$\mathscr{P}^{\mathrm{m}} = \mathbb{Q}\langle\, [M, \omega, \sigma] \,|\, M \in \mathrm{Ob}(\mathscr{M}),\ \omega \in \omega_{dR}(M),\ \sigma \in \omega_B(M)^\vee \,\rangle$$

with implicit factorisation modulo bilinearity and functoriality.

Theorem. $\mathscr{P}^{\mathrm{m}}$ is isomorphic to the space of regular functions on the $\mathbb{Q}$-scheme $\underline{\mathrm{Isom}}^{\otimes}(\omega_{dR}, \omega_B)$, that is

$$\mathscr{P}^{\mathrm{m}} \simeq \mathscr{O}(\underline{\mathrm{Isom}}^{\otimes}(\omega_{dR}, \omega_B))$$

Periods arise as a consequence of the coexistence and peculiar compatibility of the two different cohomological structures.

Galois Coaction

The motivic Galois group has a natural **action** on $\underline{\text{Isom}}^{\otimes}(\omega_{dR}, \omega_B)$

$$\nabla : G^{dR} \otimes \underline{\text{Isom}}^{\otimes}(\omega_{dR}, \omega_B) \longrightarrow \underline{\text{Isom}}^{\otimes}(\omega_{dR}, \omega_B)$$

which induces a **coaction** on $\mathcal{O}(\underline{\text{Isom}}^{\otimes}(\omega_{dR}, \omega_B)) = \mathcal{P}^{\mathfrak{m}}$

$$\begin{aligned} \Delta : \qquad \mathcal{P}^{\mathfrak{m}} &\longrightarrow \mathcal{O}(G^{dR}) \otimes \mathcal{P}^{\mathfrak{m}} \\ [M, \omega, \sigma]^{\mathfrak{m}} &\longmapsto \sum_{i=1}^{n} [M, \omega, e_i^{\vee}]^{dR} \otimes [M, e_i, \sigma]^{\mathfrak{m}} \end{aligned}$$

where $\{e_i\}$ is a basis of $\omega_{dR}(M)$ and $e_i^{\vee}$ is the dual basis.

Denote $\mathcal{P}^{dR} = \mathcal{O}(G^{dR})$.

Example

Consider $\log(z)^{\mathfrak{m}}$. The coaction gives

$$\Delta \left[M, \left[\frac{dx}{x}\right], [\gamma_1]\right]^{\mathfrak{m}} = \left[M, \left[\frac{dx}{x}\right], \left[\frac{dx}{z-1}\right]^{\vee}\right]^{dR} \otimes \left[M, \left[\frac{dx}{z-1}\right], [\gamma_1]\right]^{\mathfrak{m}}$$

$$+ \left[M, \left[\frac{dx}{x}\right], \left[\frac{dx}{x}\right]^{\vee}\right]^{dR} \otimes \left[M, \left[\frac{dx}{x}\right], [\gamma_1]\right]^{\mathfrak{m}}$$

where $M = H^1(\mathbb{G}_m, \{1, z\})$. That is

$$\Delta \log(z)^{\mathfrak{m}} = \log(z)^{dR} \otimes 1^{\mathfrak{m}} + (2\pi i)^{dR} \otimes \log(z)^{\mathfrak{m}}$$

$1^{\mathfrak{m}}$ and $\log(z)^{\mathfrak{m}}$ are the **Galois conjugates** of $\log(z)^{\mathfrak{m}}$.

Coaction Conjecture in ϕ^4 Theory

Consider the Galois coaction restricted to $\mathcal{P}^{\mathfrak{m}}_{\phi^4}$.
A priori, it has values in the whole space $\mathcal{P}^{dR} \otimes \mathcal{P}^{\mathfrak{m}}$, that is

$$\Delta : \mathcal{P}^{\mathfrak{m}}_{\phi^4} \longrightarrow \mathcal{P}^{dR} \otimes \mathcal{P}^{\mathfrak{m}}$$

Conjecture (Panzer, Schnetz). Galois conjugates of ϕ^4-periods are still ϕ^4-periods, that is

$$\Delta(\mathcal{P}^{\mathfrak{m}}_{\phi^4}) \subseteq \mathcal{P}^{dR} \otimes \mathcal{P}^{\mathfrak{m}}_{\phi^4}$$

The conjecture states the existence of a particular symmetry underlying the specific set of ϕ^4-periods.

References

- Kontsevich M. and Zagier D., *Periods*, Math. Unlimited - 2001 and Beyond, Springer, pages 771-808, 2001.
- Brown F., *Feynman Amplitudes, Action Principle, and Cosmic Galois Group*, Communications in Number Theory and Physics, 3-11, pages 453-556, 2017.
- Brown F., *Notes on Motivic Periods*, Communications in Number Theory and Physics, 3-11, pages 557-655, 2017.
- Panzer E. and Schnetz O., *The Galois Coaction on ϕ^4 Periods*, Communications in Number Theory and Physics, 3-11, pages 657-705, 2017.