Integrable systems via shifted quantum groups

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Yale

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Representation Theory and Integrable Systems
In honor of Vitaly Tarasov & Alexander Varchenko

Key Objectives

- Introduce 3^{n-2} *modified* quantum difference $\mathfrak{sl}_n$ Toda systems (joint with M. Finkelberg, 2017)

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- Generalization 1 (answering P. Etingof's question): construct $3^{\mathrm{rk}(\mathfrak{g})-1}$ *modified* quantum difference Toda systems of type $\mathfrak{g}$ (joint with R. Gonin, 2018)
- Generalization 2 (answering B. Feigin's question):
  - construct higher rank rational/trigonometric Lax matrices from **antidominantly shifted** Yangians/q.affine algebras
  - obtain Bethe subalgebras in quantized Coulomb branches
  (joint with R. Frassek and V. Pestun, 2019)

Classical and Quantum Toda systems (type A)

- *Toda lattice* is the hamiltonian system with phase space $\mathbb{R}^{2n}$ (with its usual symplectic structure) and Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_{i+1}-q_i}$$

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- **Theorem (Toda):** $\mathcal{D}_2$ defines a *quantum integrable system*:

$$\text{there exist differential operators } \{\mathcal{D}_i\}_{i=1}^n \text{ such that}$$

$$[\mathcal{D}_i, \mathcal{D}_j] = 0 \text{ and } \{\text{symbol}(\mathcal{D}_i)\}_{i=1}^n \text{ generate } \mathbb{C}[\partial_{x_1}, \ldots, \partial_{x_n}]^{\Sigma_n}$$

Lax realization of Toda systems (type A)

- Consider the *local Lax matrix*

$$L_i(z) = \begin{pmatrix} z - p_i & e^{q_i} \\ -e^{-q_i} & 0 \end{pmatrix}, \ 1 \leq i \leq n$$

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- Same for quantum Toda system with *local Lax matrices*

$$L_i(z) = \begin{pmatrix} z + \partial_{x_i} & e^{x_i} \\ -e^{-x_i} & 0 \end{pmatrix}, \ 1 \le i \le n$$

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where $q = e^{\hbar}$ and

$$T_i f(x_1, \ldots, x_n) = f(x_1, \ldots, x_i + \hbar, \ldots, x_n)$$

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- **Theorem (Ruijsenaars, '90):** There exists a family of difference operators $\{\mathcal{M}_i\}_{i=1}^{n}$ which pairwise commute and are algebraically independent.
- As $\hbar \to 0$, recover quantum Toda system.

Lax realization of quantum difference Toda (type A)

- Algebra $A_n^q = \langle \mathrm{w}_i^{\pm 1}, \mathrm{D}_i^{\pm 1} \rangle_{i=1}^n$ subject to $\mathrm{D}_i \mathrm{w}_j = q^{\delta_{ij}} \mathrm{w}_j \mathrm{D}_i$.

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- Identifying $\mathrm{w}_i \leftrightarrow T_i^{-1}, \mathrm{D}_i \leftrightarrow e^{-x_i}$, view $\mathcal{M}_i$ as elements of A_n^q.

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- Identifying $\mathrm{w}_i \leftrightarrow T_i^{-1}, \mathrm{D}_i \leftrightarrow e^{-x_i}$, view $\mathcal{M}_i$ as elements of A_n^q.
- Consider the *local Lax matrix*

$$L_i^0(z) = \begin{pmatrix} \mathrm{w}_i^{-1} z^{1/2} - \mathrm{w}_i z^{-1/2} & \mathrm{D}_i^{-1} z^{1/2} \\ -\mathrm{D}_i z^{-1/2} & 0 \end{pmatrix}, \ 1 \leq i \leq n$$

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- **Theorem (Kuznetsov-Tsyganov, '96):** The coefficients of $z^\bullet$ in $A(z)$ are the quantum difference Toda Hamiltonians.

Three Lax matrices

- In addition to $L_i^0(z)$, consider

$$L_i^{-1}(z) = \begin{pmatrix} \mathsf{w}_i^{-1} - \mathsf{w}_i z^{-1} & \mathsf{w}_i \mathsf{D}_i^{-1} \\ -\mathsf{w}_i \mathsf{D}_i z^{-1} & \mathsf{w}_i \end{pmatrix}$$

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- For

$$\vec{k} = (k_n, \ldots, k_1) \in \{-1, 0, 1\}^n,$$

consider the *mixed complete monodromy matrix*

$$\mathrm{L}_{\vec{k}}(z) := L_n^{k_n}(z) \cdots L_1^{k_1}(z) = \begin{pmatrix} A_{\vec{k}}(z) & B_{\vec{k}}(z) \\ C_{\vec{k}}(z) & D_{\vec{k}}(z) \end{pmatrix}$$

Modified quantum difference Toda systems (type A)

- **Theorem (Finkelberg-T, '17):** Fix $\vec{k} \in \{-1,0,1\}^n$.
  (a) The coefficients of $z^\bullet$ in $A_{\vec{k}}(z)$ pairwise commute.
  (b) $A_{\vec{k}}(z) = (-1)^n \mathrm{w}_1 \cdots \mathrm{w}_n \left(z^s - \mathrm{H}_2^{\vec{k}} z^{s+1} + z^{>s+1} \right)$, where $s = \sum_{j=1}^n \frac{k_j - 1}{2}$ and Hamiltonian $\mathrm{H}_2^{\vec{k}}$ equals

$$\mathrm{H}_2^{\vec{k}} = \sum_{j=1}^{n} \mathrm{w}_j^{-2} + \sum_{i=1}^{n-1} \mathrm{w}_i^{-k_i - 1} \mathrm{w}_{i+1}^{-k_{i+1}-1} \cdot \frac{\mathrm{D}_i}{\mathrm{D}_{i+1}} + \sum_{1 \leq i < j-1 \leq n-1}^{k_{i+1} = \ldots = k_{j-1} = 1} \mathrm{w}_i^{-k_i - 1} \cdots \mathrm{w}_j^{-k_j - 1} \cdot \frac{\mathrm{D}_i}{\mathrm{D}_j}$$

  (c) $\mathrm{H}_2^{\vec{k}}$ is conjugate to $\mathrm{H}_2^{\vec{k}'}$ with $\vec{k}' = (0, k_{n-1}, \ldots, k_2, 0)$.

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- This produces 3^{n-2} quantum difference Toda systems.
- For $\vec{k} = \vec{0}$, recover the above standard q-Toda.

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$$\mathcal{D}_2 = \frac{\Delta}{2} + \sum_{i=1}^{r} e^{-\alpha_i(h)},\ \Delta = \text{Laplacian}, \{\alpha_i\}_{i=1}^{r} - \text{simple roots}$$

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- **Theorem (Gonin-T, '18):** There are exactly $3^{\mathrm{rk}(\mathfrak{g})-1}$ quantum difference Toda systems, generalizing the above one. In type A, they match those obtained via 3 Lax matrices.

RLL relations for the three Lax matrices

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- Consider the trigonometric R-matrix

$$R(z,w)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{z-w}{qz-q^{-1}w} & \frac{(q-q^{-1})z}{qz-q^{-1}w} & 0 \\ 0 & \frac{(q-q^{-1})w}{qz-q^{-1}w} & \frac{z-w}{qz-q^{-1}w} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{End}\,(\mathbb{C}^2 \otimes \mathbb{C}^2)$$

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- **Proposition (Finkelberg-T, '17):** (a) The Lax matrices $L^k(z)$ $(k=-1,0,1)$ satisfy the so-called RLL relations

$$R(z,w)L_1^k(z)L_2^k(w)=L_2^k(w)L_1^k(z)R(z,w)$$

and also have quantum determinant $\mathrm{qdet}(L^k(z))=1$.
(b) Both relations also hold for $\mathrm{L}_{\vec{k}}(z)$ with $\vec{k}\in\{-1,0,1\}^n$.

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- Due to RLL and *qdet* relations, $L^k(z)$ $(k = -1, 0, 1)$ may be viewed as homomorphisms from *analogues* of $U_q(L\mathfrak{sl}_2)$ to A_1^q.

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- In the *new Drinfeld* realization of $U_q(L\mathfrak{sl}_2)$, one may shift Fourier coordinates of Cartan generators by $b^+, b^- \in \mathbb{Z}$, leading to *shifted quantum affine* $\mathfrak{sl}_2$, denoted $U_{b^+,b^-}(L\mathfrak{sl}_2)$.

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- For $a \in \mathbb{Z}_{\geq 0}$ such that $N := \frac{1}{2}(a - b^+ - b^-) \in \mathbb{Z}_{>0}$, there are distinguished homomorphisms

$$\Phi^a_{b^+,b^-} : U_{b^+,b^-}(L\mathfrak{sl}_2) \to A_N^q$$

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- In the *new Drinfeld* realization of $U_q(L\mathfrak{sl}_2)$, one may shift Fourier coordinates of Cartan generators by $b^+, b^- \in \mathbb{Z}$, leading to *shifted quantum affine* $\mathfrak{sl}_2$, denoted $U_{b^+,b^-}(L\mathfrak{sl}_2)$.
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- **Theorem (Finkelberg-T, '17):** (a) The above three *analogues* of $U_q(L\mathfrak{sl}_2)$ are isomorphic to the shifted quantum affine $U_{0,-2}(L\mathfrak{sl}_2), U_{-1,-1}(L\mathfrak{sl}_2), U_{-2,0}(L\mathfrak{sl}_2)$.
(b) The above homomorphisms from these three *analogues* to A_1^q coincide with $\Phi^0_{0,-2}, \Phi^0_{-1,-1}, \Phi^0_{-2,0}$.

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- Our motivation comes from Frassek-Pestun (2018):
  Input: a pair of Young diagrams $\boldsymbol{\lambda}, \boldsymbol{\mu}$ of total size n and a collection of points $\{x_i\}$–one for each column of $\boldsymbol{\lambda}$
  Output: a rational Lax matrix $L_{\boldsymbol{\lambda},\underline{x},\boldsymbol{\mu}}(z)$ linear in z with $\mathrm{qdet}\ L_{\boldsymbol{\lambda},\underline{x},\boldsymbol{\mu}}(z) = \prod_i \prod_{k=1}^{\boldsymbol{\lambda}_i^t} (z - x_i - (k-1))$.

Higher rank generalization

- Question: Can we generalize this to $n \times n$ Lax matrices?
- Related literature:
  - Physics: V. Bazhanov *et. al.*, Z. Tsuboi, ...
  - Mathematics: Hernandez-Jimbo (cf. Felder-Zhang)
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- Their Questions:
  - Degeneration procedure (moving columns from $\boldsymbol{\lambda}$ to $\boldsymbol{\mu}$)
  - Trigonometric counterpart (depending on 3 Young diagrams)
  - Degeneration of trigonometric Lax to rational Lax

Higher rank generalization

- **Theorem (Frassek-Pestun-T, '19):** (a) The shifted Yangian $Y_\mu(\mathfrak{gl}_n)$ admits an RLL realization iff μ–**antidominant**.
(b) Homomorphisms of [BFN, '16] produce rational Lax matrices $\mathrm{L}_{\boldsymbol{\lambda},\underline{x};\boldsymbol{\mu}}(z)$ polynomial in z of degree $\frac{|\boldsymbol{\lambda}|+|\boldsymbol{\mu}|}{n}$ for any pair of Young diagrams of length $< n$ and $n||\boldsymbol{\lambda}|+|\boldsymbol{\mu}|$.
(c) $\mathrm{L}_{\boldsymbol{\lambda},\underline{x};\boldsymbol{\mu}}(z)$ is a normalized limit of $\mathrm{L}_{\boldsymbol{\lambda}\cup\boldsymbol{\mu},\underline{x}\cup\underline{x}';\emptyset}(z), x'_? \to \infty$.

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- Also $\mathrm{L}_{\boldsymbol{\lambda},\underline{x};\boldsymbol{\mu}^+,\boldsymbol{\mu}^-}(z)$ degenerates to $\mathrm{L}_{\boldsymbol{\lambda},\underline{x};\boldsymbol{\mu}^+\cup\boldsymbol{\mu}^-}(z)$.

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- Homomorphisms $Y_{\mu_1+\mu_2}(L\mathfrak{gl}_n) \to Y_{\mu_1}(L\mathfrak{gl}_n) \otimes Y_{\mu_2}(L\mathfrak{gl}_n)$, recovering the main construction of [FKPRW, '16].

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- Recover coproduct homomorphisms of [Finkelberg-T, '17]: $U_{\mu_1^+ + \mu_2^+, \mu_1^- + \mu_2^-}(L\mathfrak{gl}_n) \to U_{\mu_1^+, \mu_1^-}(L\mathfrak{gl}_n) \otimes U_{\mu_2^+, \mu_2^-}(L\mathfrak{gl}_n)$.

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- Obtain Bethe subalgebras $B(C)$ for $C \in \mathrm{Mat_{n\times n}}(\mathbb{C})$.
- By the construction of [BFN, '16] and [Finkelberg-T, '17], actually obtain Bethe subalgebras in quantized (K-theoretic) Coulomb branches of type A quiver gauge theories

References

This talk is based on the recent joint works with Michael Finkelberg, Roman Gonin, Rouven Frassek, Vasily Pestun:

[Finkelberg-T, '17] *Multiplicative slices, relativistic Toda and shifted quantum affine algebras*, Progress in Mathematics (2019), 172pp, DOI:10.1007/978-3-030-23531-4.

[Finkelberg-T, '18] *Shifted quantum affine algebras: integral forms in type A*, Arnold Mathematical Journal (2019), 87pp, DOI:10.1007/s40598-019-00118-7.

[Gonin-T, '18] *On Sevostyanov's construction of quantum difference Toda lattices*, International Mathematics Research Notices (2019), 61pp, DOI:10.1093/imrn/rnz083.

[Frassek-Pestun-T, '19] *Rational and trigonometric Lax matrices via antidominantly shifted Yangians and quantum affine algebras*, in preparation.

The End

Thank you!