Duality for Bethe algebras acting on polynomials in anticommuting variables

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Space $\mathfrak{P}_{kn}$

$\mathfrak{P}_{kn}$ – space of polynomials in ξ_{ai}, $a = 1, \dots, k$, $i = 1, \dots, n$.
$\xi_{ai}\xi_{bj} = -\xi_{bj}\xi_{ai}$, $(a,i) \neq (b,j)$, $\xi_{ai}^2 = 0$ for any a, i.

The left derivations ∂_{ai}, $a = 1, \dots, k$, $i = 1, \dots, n$:
For monomial $g \in \mathfrak{P}_{kn}$ such that $\xi_{ai} g \neq 0$, we have:
$\partial_{ai} g = 0$, $\partial_{ai}(\xi_{ai} g) = g$.

Fix $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \neq \alpha_j$, $i \neq j$. Define $\mathfrak{gl}_k[t]$-action on $\mathfrak{P}_{kn}$:

$$\pi_{\bar{\alpha}}^{\langle k \rangle} : \quad e_{ab}^{\langle k \rangle} \otimes t^s \mapsto \sum_{i=1}^{n} \alpha_i^s \xi_{ai} \partial_{bi}.$$

Fix $\bar{z} = (z_1, \dots, z_k)$, $z_a \neq z_b$, $a \neq b$. Define $\mathfrak{gl}_n[t]$-action on $\mathfrak{P}_{kn}$:

$$\pi_{\bar{z}}^{\langle n \rangle} : \quad e_{ij}^{\langle n \rangle} \otimes t^s \mapsto \sum_{a=1}^{k} z_a^s \xi_{ai} \partial_{aj}.$$

Bethe algebras

Let $e_{ab}^{\langle k\rangle}(x)=\sum_{s=0}^{\infty}(e_{ab}^{\langle k\rangle}\otimes t^s)x^{-s-1}$. Consider

$$\operatorname{cdet}\left(\delta_{ab}\left(\frac{d}{dx}-z_a\right)-e_{ab}^{\langle k\rangle}(x)\right)_{a,b=1}^{k}=$$

$$=\left(\frac{d}{dx}\right)^k+\sum_{a=1}^{k}\left(\sum_{b=0}^{\infty}B_{ab}^{\langle k\rangle}x^{-b}\right)\left(\frac{d}{dx}\right)^{k-a}.$$

The Bethe algebra $\mathcal{B}_{\bar{z}}^{\langle k\rangle}\subset U(\mathfrak{gl}_k[t])$ is the subalgebra generated by $B_{ab}^{\langle k\rangle}$, $a=1,\dots,k,\ b\geq 0$.

Similarly, define $\mathcal{B}_{\bar{\alpha}}^{\langle n\rangle}\subset U(\mathfrak{gl}_n[t])$. Denote the corresponding generators $B_{ij}^{\langle n\rangle}$, $i=1,\dots,n$, $j\geq 0$.

Theorem ([Huang, Mukhin], [Tarasov, U.])

$$\pi_{\bar{z}}^{\langle n\rangle}(\mathcal{B}_{\bar{\alpha}}^{\langle n\rangle})=\pi_{-\bar{\alpha}}^{\langle k\rangle}(\mathcal{B}_{\bar{z}}^{\langle k\rangle}).$$

Spaces of quasi-exponentials

Fix $\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, $\alpha_i \neq \alpha_j$, $i \neq j$, $\bar{\mu} = (\mu^{(1)}, \dots, \mu^{(n)})$, $\mu^{(i)} = (\mu_1^{(i)}, \mu_2^{(i)}, \dots, \mu_{n_i}^{(i)}, 0, 0, \dots, 0, \dots)$, $\mu_1^{(i)} \geq \mu_2^{(i)} \geq \dots \geq \mu_{n_i}^{(i)} > 0$.
Assume $n_i > 0$.

Let V be a space of functions with a basis of the form

$$\{e^{\alpha_i x} p_{ij}(x) |\ i = 1, \dots, n,\ j = 1, \dots, n_i\},$$

where $p_{ij}(x)$ are polynomials and $\deg p_{ij} = n_i + \mu_j^{(i)} - j$.

Let $\mathbf{e}(z) = (e_1(z) > e_2(z) > \dots > e_n(z))$ be exponents of V at a point $z \in \mathbb{C}$, that is for each $i = 1, \dots, n$, there is $f(x) \in V$ such that $f(x) = (x - z)^{e_i(z)}(1 + o(1))$.
Define a partition $\lambda(z)$ by the rule: $e_i(z) = \dim V + \lambda_i(z) - i$.

Spaces of quasi-exponentials

A point $z \in \mathbb{C}$ is called singular if $\lambda(z) \neq (0, 0, 0, \dots)$.
Let $\{z_1, \dots, z_k\}$ be the set of all singular points of V.
Denote $\lambda(z_a) = \lambda^{(a)}$, $\bar{z} = (z_1, \dots, z_k)$, $\bar{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$.

We say that V is a **space of quasi-exponentials with the data** $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$.

The fundamental differential operator D_V **of** V is the unique monic differential operator of order $\dim V$ such that $D_V f = 0$ for any $f \in V$.

Transformation $D_V \to \tilde{D}_V$

- Define the transformation $D \to D^\dagger$ of differential operators by:

  $(\cdot)^\dagger$ is an antiautomorphism,
  $\left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}$,
  $(b(x))^\dagger = b(x)$.

  $D^\dagger$ is called the **formal conjugate** of D.

- Define the transformation $D \to D^\ddagger$ of differential operators with polynomial coefficients by:

  $(\cdot)^\ddagger$ is an antiautomorphism,
  $\left(\frac{d}{dx}\right)^\ddagger = x$,
  $x^\ddagger = \frac{d}{dx}$.

  $D^\ddagger$ is called the **bispectral dual** of D.

Transformation $D_V \to \tilde{D}_V$

There exists differential operator $\check{D}_V$ such that

$$\prod_{i=1}^{n}\left(\frac{d}{dx}-\alpha_i\right)^{n_i+\mu_1^{(i)}} = \check{D}_V D_V.$$

Consider a chain of transformations:

$$D_V \to \check{D}_V \to \check{D}_V^{\dagger} \to (p\check{D}_V^{\dagger})^{\ddagger},$$

where p is the polynomal of minimal degree such that $p\check{D}_V^{\dagger}$ is a differential operator with polynomial coefficients.

Theorem ([Tarasov, U.])

The space $\tilde{V} = \ker((p\check{D}_V^{\dagger})^{\ddagger})$ is a space of quasi-exponentials with the data $(\bar{\lambda}', \bar{\mu}'; \bar{z}, -\bar{\alpha})$.

Let $\tilde{D}_V$ be the fundamental differential operator of $\tilde{V}$.

Weight subspaces $\mathfrak{P}_{kn}[\lambda, \mu]$

Fix $\lambda = (l_1, \dots, l_k)$, $l_a \in \mathbb{Z}_{>0}$, $a = 1, \dots, k$,
$\mu = (m_1, \dots, m_n)$, $m_i \in \mathbb{Z}_{>0}$, $i = 1, \dots, n$.
Consider a subspace $\mathfrak{P}_{kn}[\lambda, \mu] \subset \mathfrak{P}_{kn}$,

$$\mathfrak{P}_{kn}[\lambda, \mu] = \{p \in \mathfrak{P}_{kn} | \ e_{aa}^{\langle k \rangle} p = l_a p, \ e_{ii}^{\langle n \rangle} p = m_i p\}.$$

Both $\mathcal{B}_{\bar{z}}^{\langle k \rangle}$ and $\mathcal{B}_{\bar{\alpha}}^{\langle n \rangle}$ preserve the subspace $\mathfrak{P}_{kn}[\lambda, \mu]$.

Eigenvectors for Bethe algebras

Define $\bar{\mu} = (\mu^{(1)}, \dots, \mu^{(n)})$ and $\bar{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$ by:
$\mu^{(i)} = (m_i, 0, 0, \dots)$, $\lambda^{(a)} = (\underbrace{1, \dots, 1}_{l_a}, 0, 0, \dots)$.

Theorem ([Mukhin, Tarasov, Varchenko])

- *There is a bijective correspondence:*

$$\left\{ \begin{array}{l} \textit{eigenvectors of} \\ \pi^{\langle n \rangle}_{\bar{z}}(\mathcal{B}^{\langle n \rangle}_{\bar{\alpha}}) \textit{ in } \mathfrak{P}_{kn}[\lambda, \mu] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textit{spaces of quasi-exponentials} \\ \textit{with the data } (\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z}) \end{array} \right\}$$

- *Similarly, there is a bijective correspondence:*

$$\left\{ \begin{array}{l} \textit{eigenvectors of} \\ \pi^{\langle k \rangle}_{-\bar{\alpha}}(\mathcal{B}^{\langle k \rangle}_{\bar{z}}) \textit{ in } \mathfrak{P}_{kn}[\lambda, \mu] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textit{spaces of quasi-exponentials} \\ \textit{with the data } (\bar{\lambda}', \bar{\mu}'; \bar{z}, -\bar{\alpha}) \end{array} \right\}$$

Duality and spaces of quasi-exponentials

Let $v \in \mathfrak{P}_{kn}[\lambda, \mu]$ be an eigenvector of $\pi^{\langle n \rangle}_{\bar{z}}(\mathcal{B}^{\langle n \rangle}_{\bar{\alpha}})$, and let V be the corresponding space of quasi-exponentials.

Notice that $\tilde{V}$ corresponds to an eigenvector of $\pi^{\langle k \rangle}_{-\bar{\alpha}}(\mathcal{B}^{\langle k \rangle}_{\bar{z}})$ in $\mathfrak{P}_{kn}[\lambda, \mu]$.

Theorem ([Tarasov, U.])

The vector v is the eigenvetor of $\pi^{\langle k \rangle}_{-\bar{\alpha}}(\mathcal{B}^{\langle k \rangle}_{\bar{z}})$ corresponding to $\tilde{V}$.

Eigenvalues of Bethe algebras

Let $b_i(x)$ be the coefficients of D_V:

$$D_V = \left(\frac{d}{dx}\right)^n + \sum_{i=1}^{n} b_i(x) \left(\frac{d}{dx}\right)^{n-i}.$$

Let $\sum_{j=0}^{\infty} b_{ij} x^{-j}$ be the Laurent series of $b_i(x)$ at infinity.

Recall that the Bethe algebra $\mathcal{B}^{\langle n \rangle}_{\bar{\alpha}}$ is generated by $B^{\langle n \rangle}_{ij}$, $i = 1, \ldots, n$, $j \geq 0$.

Theorem ([Mukhin, Tarasov, Varchenko])

The eigenvalue of $B^{\langle n \rangle}_{ij}$ associated to eigenvector v is b_{ij}.

- We can express coefficients of $\tilde{D}_V$ in terms of coefficients of D_V $\longrightarrow$ we know how the eigenvalues of two Bethe algebras are linked.
- These expressions lift to expressions for generators, which gives the duality.

Thank You!