Hypergeometric Integrals of Motion and Affine Gaudin Models

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based on work with **Sylvain Lacroix** and **Benoît Vicedo**

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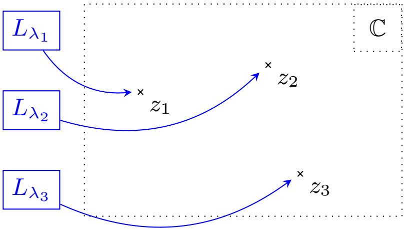
Quantum Gaudin Model

- Let $\mathfrak{g}$ be any symmetrizable Kac-Moody algebra
- Assign irreducible highest-weight $\mathfrak{g}$-modules $\{L_{\lambda_i}\}$ to marked points $\{z_i\}$ in $\mathbb{C}$:



- Canonical element: $\Xi = \sum_{\alpha} \Xi_\alpha, \quad \Xi_\alpha \in \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$

$$\mathcal{H}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\Xi^{(ij)}}{z_i - z_j} \in U(\mathfrak{g})^{\otimes N}, \qquad i = 1, \dots, N$$

Quadratic Gaudin Hamiltonians. Realize as maps $\mathcal{H}_i : \bigotimes_{k=1}^{N} L_{\lambda_k} \to \bigotimes_{k=1}^{N} L_{\lambda_k}$

Bethe ansatz for Gaudin models

- Gaudin model solvable by a form of Bethe ansatz:
  - Pick $m \geq 0$ additional marked points t_j (Bethe roots)
  - Associate to each a simple root $\alpha_{c(j)}$
  - Construct Bethe vector $\psi = \psi(\{z_i\}, \{\lambda_i\}, \{t_j\}, \{\alpha_{c(j)}\})$
- **Theorem**: if Bethe roots $\{t_j\}$ obey Bethe equations then ψ is a joint eigenvector of the $\mathcal{H}_i$, with explicit eigenvalues.

$$-\sum_{i=1}^{N} \frac{(\lambda_i|\alpha_{c(j)})}{t_j - z_i} + \sum_{\substack{i=1 \\ i\neq j}}^{m} \frac{(\alpha_{c(i)}|\alpha_{c(j)})}{t_j - t_i} = 0, \quad j = 1, \dots, m.$$

$$E_i := \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{(\lambda_i|\lambda_j)}{z_i - z_j} - \sum_{j=1}^{m} \frac{(\lambda_i|\alpha_{c(j)})}{z_i - t_j}, \qquad i = 1, \dots, N.$$

- Theorem holds for **any** symmetrizable Kac-Moody algebra $\mathfrak{g}$.
  (General proof is in terms of hyperplane arrangements [Schechtman & Varchenko, '91])

Finite type $\mathfrak{g}$ – Gaudin algebra and Opers

For $\mathfrak{g}$ of finite type much more is known:

$$\mathcal{H}_i \in \mathscr{B} \subset U(\mathfrak{g})^{\otimes N}$$

Bethe Algebra

- $\mathscr{B}$: a (large) commutative subalgebra of $U(\mathfrak{g})^{\otimes N}$ generated by $\mathcal{H}_i$ together with higher Gaudin Hamiltonians

  [Feigin Frenkel Reshetikhin]

  (for which explicit formulas exist [Talalaev] [Molev])
- ψ is a joint eigenvector for the entire algebra $\mathscr{B}$

  [Feigin Frenkel Reshetikhin]
- Joint eigenvalues encoded as functions on a space of opers
- (Completeness of Bethe ansatz)
  For integral dominant highest weights λ_i, can identify image of $\mathscr{B}$ in $\mathrm{End} \bigotimes_{i=1}^{N} L_{\lambda_i}$ with algebra of functions on a certain well-defined space of monodromy-free opers.

  [Mukhin Tarasov Varchenko '09] (type A) [Rybnikov '18] (all finite types)

Main questions:

Suppose $\mathfrak{g}$ is of untwisted affine type

1. Are there higher Gaudin Hamiltonians?
2. If yes, then what parameterizes their eigenvalues?
   Functions on opers? What opers? What do such functions look like?

...important questions for mathematical physics because affine (quantum) Gaudin models are closely related to integrable (quantum) field theories in $1 + 1$ dimensions

Plan of this talk:

(i) Define a notion of affine opers, generalizing definitions from finite type in the most direct way possible.

(ii) Main result: the functions on the space of affine opers are of a very different character than in the finite case:
they are given by hypergeometric-type integrals over cycles of a twisted homology defined by the levels of the modules at the marked points.

(iii) Conjecture: these integrals are the eigenvalues of (higher) Gaudin Hamiltonians
(...prompts a conjecture about the form of such Hamiltonians themselves)

(iv) Check this conjecture in some special cases

Review: Opers and Miura opers in finite types

Suppose $\mathfrak{g}$ is of finite type. Let ${}^L\mathfrak{g}$ be its Langlands dual (also of finite type).

- Cartan decomposition: ${}^L\mathfrak{g} = {}^L\mathfrak{n}_- \oplus {}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_+$
- Chevalley generators: $\check{f}_i$, $\check{e}_i$, $i = 1, \ldots, \ell$.
- Simple coroots: $\alpha_i := [\check{e}_i, \check{f}_i]$ (are the simple roots of $\mathfrak{g}$)

Definition: A <u>**Miura ${}^L\mathfrak{g}$-oper**</u> is a connection of the form [Frenkel]

$$\nabla = d + \Big(p_{-1} \ + \ u(z) \Big) dz.$$

Principal nilpotent element
$p_{-1} := \sum_{i=1}^{\ell} \check{f}_i$

rational function valued in Cartan ${}^L\mathfrak{h} \cong \mathfrak{h}^*$

For us, $u(z)$ is of the form

$$u(z) = -\sum_{i=1}^{N} \frac{\lambda_i}{z - z_i} + \sum_{j=1}^{m} \frac{\alpha_{c(j)}}{z - t_j}$$

and encodes the marked points $\{z_i\}$, Bethe roots $\{t_j\}$, highest weights $\{\lambda_i\}$ and "colours" of the Bethe roots $\{c(j)\}$.

Definition: An **${}^L\mathfrak{g}$-oper** is a **gauge equivalence class** $[\nabla]$ of connections of the form

$$\nabla = d + \Big(p_{-1} \; + \; b(z) \Big) \, dz$$

rational function valued in Borel ${}^L\mathfrak{b}_+ \cong {}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_+$

under the gauge action of the unipotent subgroup ${}^L N = \exp({}^L\mathfrak{n}_+)$.

Fact: Each oper $[\nabla]$ has a unique representative of **canonical form**

$$d + \left(p_{-1} + \sum_{r \in E} \; v_r(z) \;\; p_r \right) dz.$$

finite (multi)set of exponents of ${}^L\mathfrak{g}$

rational coefficient functions

element $p_r \in {}^L\mathfrak{n}_+$ of grade r in principal gradation (h.w. vector for principal $\mathfrak{sl}_2$)

Corollary: These $v_r(z)$ are "good coordinates" on the space of opers.

- Each Miura oper ∇ defines an underlying oper $[\nabla]$

Fact: The Bethe equations precisely ensure $v_r(z)$ have poles only at the marked points $z_1, \ldots, z_N$ (and ∞) and not at the Bethe roots $t_1, \ldots, t_m$.

Dictionary:

Miura oper $\nabla \longleftrightarrow \quad u(z) \in {}^L\mathfrak{h} \quad \longleftrightarrow$ joint eigenvector ψ of Gaudin Hamiltonians

Underlying oper $[\nabla] \longleftrightarrow \{v_r(z) \in \mathbb{C}\}_{r \in E} \longleftrightarrow$ eigenvalues of all Gaudin Hamiltonians

Opers and Miura opers in affine types

Suppose $\mathfrak{g}$ is of untwisted affine type. Let ${}^L\mathfrak{g}$ be Langlands dual (affine, maybe twisted).

- Cartan decomposition: ${}^L\mathfrak{g} = {}^L\mathfrak{n}_- \oplus {}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_+$
- Chevalley generators: $\check{f}_i$, $\check{e}_i$, $i = 0, 1, \ldots, \ell$; coroots $\alpha_i = [\check{e}_i, \check{f}_i]$

Definition: A **Miura ${}^L\mathfrak{g}$-oper** is a connection of the form

$$\nabla = d + \Big(p_{-1} \ + \ u(z) \Big) dz.$$

Principal ~~nilpotent~~ element
$p_{-1} := \sum_{i=0}^{\ell} \check{f}_i$

rational function valued in Cartan ${}^L\mathfrak{h} \cong \mathfrak{h}^*$

- $u(z)$ as before – except 'colours' of Bethe roots $c(j) \in \{0, 1, \ldots, \ell\}$ can include 0.
- Principal derivation element: $\rho \in {}^L\mathfrak{h}$. $[\rho, \check{e}_i] = \check{e}_i$, $[\rho, \check{f}_i] = -\check{f}_i$.
- Decompose $u(z)$ in basis $\{\alpha_i\}_{i=0}^{\ell} \cup \{\rho\}$:

$$\nabla = d + \left(p_{-1} - \frac{\varphi(z)}{h^\vee}\rho + \sum_{i=0}^{\ell} u_i(z)\alpha_i \right) dz, \qquad \varphi(z) = \sum_{i=1}^{N} \frac{k_i}{z - z_i}$$

where $k_i = \langle \mathsf{k}, \lambda_i \rangle$ are the levels of the L_{λ_i}. Call $\varphi(z)$ the twist function.

Definition: An ${}^L\mathfrak{g}$**-oper** is a **gauge equivalence class** $[\nabla]$ of connections of the form

$$\nabla = d + \Big(p_{-1} \ + \ b(z) \Big)\, dz$$

rational function valued in Borel ${}^L\mathfrak{b}_+ \cong {}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_+$

under the gauge action of the ~~unipotent~~ subgroup ${}^L N = \exp({}^L\mathfrak{n}_+)$.

Theorem: [Lacroix, Vicedo, CY] (following [Drinfeld Sokolov])
(i) Each oper $[\nabla]$ has a ~~unique~~ representative of **quasi-canonical form**

$$d + \left(p_{-1} - \frac{\varphi(z)}{h^\vee}\rho + \sum_{r \in E} v_r(z)\, p_r \right) dz.$$

countably infinite (multi)set of exponents of ${}^L\mathfrak{g}$

rational coefficient functions

element $p_r \in {}^L\mathfrak{n}_+$ of
grade r in principal gradation
($\in$ principal Heisenberg subalgebra)

(ii) The functions $\varphi(z)$ and $v_1(z)$ are unique. But the functions $v_r(z)$, $r \geq 2$, are unique only up to transformations of the form

$$v_r(z) \longmapsto v_r(z) - g_r'(z) + \frac{r\varphi(z)}{h^\vee} g_r(z)$$

for any rational functions $g_r(z)$.

~~**Corollary:** These $v_r(z)$ are "good coordinates" on the space of affine opers.~~

. . . so how to construct well-defined functions on the space of affine opers?

- Define multivalued function $\mathcal{P}(z) := \prod_{i=1}^{N} (z - z_i)^{k_i}$ whose log-derivative is $\varphi(z)$.
- Gauge freedom in $v_r(z)$ is then

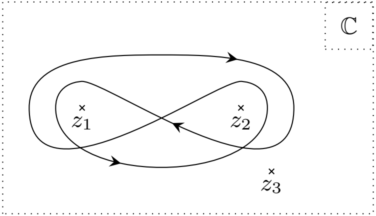
$$\mathcal{P}(z)^{-r/h^\vee} v_r(z) \longmapsto \mathcal{P}(z)^{-r/h^\vee} v_r(z) - \partial_z \left(\mathcal{P}(z)^{-r/h^\vee} g_r(z) \right).$$

- To get gauge-invariant quantities we should consider integrals of $\mathcal{P}(z)^{-r/h^\vee} v_r(z) \ldots$

integrals over any cycle γ which is not only closed but also around which $\mathcal{P}$ is single-valued...

$$I_\gamma^{(r)} := \int_\gamma \mathcal{P}(z)^{-r/h^\vee} v_r(z) dz$$

Prototypical example of such cycles are Pochhammer contours



Corollary: These integrals $I_\gamma^{(r)}$ are well-defined functions on the space of affine opers.

Proposition: The Bethe equations precisely ensure there exists a gauge in which $\{v_r(z)\}_{r\in E}$ have poles only at the marked points $z_1, \ldots, z_N$ (and ∞) and not at the Bethe roots $t_1, \ldots, t_m$.

Aside: Coordinate-independent statements

For ${}^L\mathfrak{g}$ of finite type,

$$\mathrm{Op}_{{}^L\mathfrak{g}}(U) \simeq \mathrm{Proj}(U) \times \prod_{j \in E_{\geq 2}} \Gamma(U, \Omega^{j+1}),$$

[Frenkel]

Theorem [Lacroix,Vicedo,CY]
For ${}^L\mathfrak{g}$ of affine type, $\mathrm{Op}_{{}^L\mathfrak{g}}(U)$ fibres over $\mathrm{Conn}(U, \Omega)$ and

$$\mathrm{Op}_{{}^L\mathfrak{g}}(U)^{\nabla} \simeq \Gamma(U, \Omega^2) \times \prod_{j \in E_{\geq 2}} H^1(U, \Omega^j, \nabla)$$

where $\mathrm{Op}_{{}^L\mathfrak{g}}(U)^{\nabla}$ is the fibre over a connection $\nabla \in \mathrm{Conn}(U, \Omega)$. $\square$

Conjectures

1. These integrals $I_\gamma^{(r)}$ are the eigenvalues of higher affine Gaudin Hamiltonians.
2. The Hamiltonians themselves are integrals,

$$H_\gamma^{(r)} := \int_\gamma \mathcal{P}(z)^{-r/h^\vee} S_r(z)_0 \, dz$$

for certain "densities" $S_r(z)_0 \in \hat{U}(\mathfrak{g}^{\oplus N})$ depending rationally on z.

In particular, each Hamiltonian is labelled by

- an exponent r from the infinite multiset E of exponents and
- a cycle γ of the twisted homology

Checks

- Semiclassics
- Cubic Hamiltonians
- GKO coset constructions (2-point Gaudin models for $\widehat{\mathfrak{sl}_2}$ and $\widehat{\mathfrak{sl}_3}$)

Semiclassics

Recall results on classical Principal Chiral Models (PCMs)

[Evans, Hassan, MacKay, Mountain]

- Let $j_+ = g^{-1}\partial_+ g$ where $g = g(x,t) \in G$ is the PCM field.
- There are Poisson-commuting conserved charges of the form

$$\int_0^{2\pi} dx K_{ab\ldots c} j_+^a j_+^b \ldots j_+^c.$$

  Here $K_{ab\ldots c}$ are certain invariant tensors whose degrees $\in$ { exponents of G repeating modulo the Coxeter number } = { the exponents of the affine algebra }
- Classical PCMs can be interpreted as classical affine Gaudin models and then these conserved charges are of the form

[Vicedo],[Lacroix, Magro, Vicedo]

$$\int_0^{2\pi} dx K_{ab\ldots c} L(z_{(0)})^a L(z_{(0)})^b \ldots L(z_{(0)})^c.$$

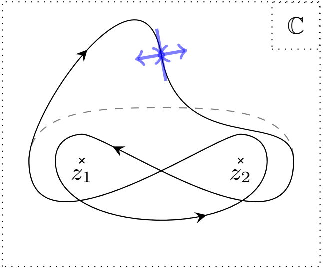
  where $L(z)$ is the (Gaudin) Lax matrix and $z_{(0)}$ is a zero of the twist function $\varphi(z)$.

Semiclassics

On the other hand, one can re-introduce $\hbar$ in the quantum-mechanical constructions above:

$$H_\gamma^{(r)} = \int_\gamma \mathcal{P}(z)^{-r/(\hbar h^\vee)} S_r^{(\hbar)}(z)_0 dz$$

Then in the $\hbar \to 0$ limit, deform contour γ to apply method of steepest descents:



Integrals of the form $H_\gamma^{(r)}$ localize at the saddle points of $\mathcal{P}(z)$ = zeros of $\varphi(z)$!

(And count of zeros $(= N - 1)$ agrees with count of independent cycles.)

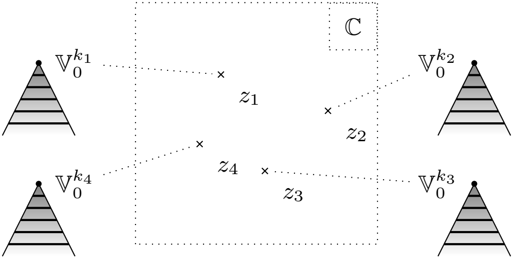
(Reminiscent of passage from KZ equations to Gaudin model – yet conceptually quite separate)

Cubic Hamiltonians

- Simplest general direct check is in types $\widehat{\mathfrak{sl}}_M$ with $M \geq 3$.
- Check for $r = 1, 2$ only so far, i.e. quadratic and cubic Hamiltonians.
- (Guess that) densities $S_r(z)_0$ are actually Fourier **zero modes** of certain **states** in tensor product of Vacuum verma modules $\mathbb{V}_0^{\boldsymbol{k}} = \bigotimes_{i=1}^N \mathbb{V}_0^{k_i}$

$$S_1(z) := \tfrac{1}{2} I^a_{-1}(z) I^a_{-1}(z) \left|0\right\rangle^{\boldsymbol{k}} ,$$
$$S_2(z) := \tfrac{1}{3} t_{abc} I^a_{-1}(z) I^b_{-1}(z) I^c_{-1}(z) \left|0\right\rangle^{\boldsymbol{k}} ,$$

where $I^a_{-1}(z) := \sum_{i=1}^N \frac{I_{-1}^{a,\text{(tensor factor i)}}}{z - z_i}$ and $\left|0\right\rangle^{\boldsymbol{k}} = \left|0\right\rangle \otimes \ldots \otimes \left|0\right\rangle .$



Theorem: [Lacroix,Vicedo,CY] For $i, j \in \{1, 2\}$,

$$S_i(z)_{(0)}S_j(w) = D_z^{(i)}A_{ij}(z,w) + D_w^{(j)}B_{ij}(z,w) + TC_{ij}(z,w),$$

for some $\mathbb{V}_0^{\boldsymbol{k}}$-valued rational functions $A_{ij}(z,w)$, $B_{ij}(z,w)$ and $C_{ij}(z,w)$.
Proof Direct (lengthy) calculation... e.g.

$$\begin{aligned}
A_{22}(z,w) = \Bigg(&\frac{2h^{\vee 3}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^3} I^a_{-4}(z)I^a_{-1}(z) - \frac{4h^{\vee 3}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^3} I^a_{-4}(z)I^a_{-1}(w) \\
&- \frac{2h^{\vee 2}\left(1-\frac{4}{h^{\vee 2}}\right)\varphi(z)}{(z-w)^2} I^a_{-4}(z)I^a_{-1}(w) - \frac{h^{\vee 3}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^2} I^a_{-4}{}'(z)I^a_{-1}(w) \\
&- \frac{2h^{\vee 3}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^3} I^a_{-3}(z)I^a_{-2}(z) + \frac{2h^{\vee 2}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^2} f_{abc}I^a_{-3}(z)I^b_{-1}(z)I^c_{-1}(w) \\
&+ \frac{h^\vee}{z-w} t_{abe}t_{cde}I^a_{-2}(z)I^b_{-1}(z)I^c_{-1}(w)I^d_{-1}(w) \Bigg) |0\rangle^{\boldsymbol{k}}
\end{aligned}$$

$$\begin{aligned}
B_{22}(z,w) = \Bigg(&- \frac{2h^{\vee 3}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^3} I^a_{-4}(z)I^a_{-1}(z) + \frac{8h^{\vee 3}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^3} I^a_{-4}(z)I^a_{-1}(w) \\
&+ \frac{4h^{\vee 2}\left(1-\frac{4}{h^{\vee 2}}\right)\varphi(z)}{(z-w)^2} I^a_{-4}(z)I^a_{-1}(w) - \frac{2h^{\vee 2}\left(1-\frac{4}{h^{\vee 2}}\right)\varphi(w)}{(z-w)^2} I^a_{-4}(z)I^a_{-1}(w) \\
&+ \frac{h^{\vee 3}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^2} I^a_{-4}(z)I^a_{-1}{}'(w) + \frac{2h^{\vee 3}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^3} I^a_{-3}(z)I^a_{-2}(z) \\
&- \frac{2h^{\vee 2}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^2} f_{abc}I^a_{-3}(z)I^b_{-1}(z)I^c_{-1}(w) \\
&- \frac{h^\vee}{z-w} t_{abe}t_{cde}I^a_{-2}(z)I^b_{-1}(z)I^c_{-1}(w)I^d_{-1}(w) \Bigg) |0\rangle^{\boldsymbol{k}}
\end{aligned}$$

$$\begin{aligned}
C_{22}(z,w) = \Bigg(&\frac{2h^{\vee 3}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^4} I^a_{-3}(z)I^a_{-1}(z) \\
&+ \frac{\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^3}\left(-4h^{\vee 2}\varphi(z) + 4h^{\vee 2}\varphi(w) - 10\frac{h^{\vee 3}}{z-w}\right) I^a_{-3}(z)I^a_{-1}(w) \\
&- \frac{2h^{\vee 3}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^3} I^a_{-3}(z)I^a_{-1}{}'(w) - \frac{2h^{\vee 3}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^4} I^a_{-2}(z)I^a_{-2}(z) \\
&+ \frac{h^{\vee 3}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^4} I^a_{-2}(z)I^a_{-2}(w) + \frac{2h^{\vee 2}\left(1-\frac{4}{h^{\vee 2}}\right)}{(z-w)^3} f_{abc}I^a_{-2}(z)I^b_{-1}(z)I^c \\
&+ \frac{h^\vee}{2(z-w)^2} t_{abe}t_{cde}I^a_{-1}(z)I^b_{-1}(z)I^c_{-1}(w)I^d_{-1}(w) \Bigg) |0\rangle^{\boldsymbol{k}}.
\end{aligned}$$

Corollary: The corresponding Hamiltonians, i.e. contour integrals of zero modes, commute.

GKO coset construction and qKdV integrals of motion

Consider Gaudin model for $\widehat{\mathfrak{sl}}_2$ with 2 marked points.
Quadratic Hamiltonian:

$$\mathcal{H} := \mathcal{H}_1 = -\mathcal{H}_2 = \frac{\Xi}{z_1 - z_2} \quad \text{where} \quad \Xi = d \otimes k + k \otimes d + \sum_n I^a_n \otimes I_{a,-n}$$

On the other hand, have Segal-Sugawara generators of Virasoro algebra at sites 1 and 2, and the diagonal copy:

$$T^{(1)}(x) := \frac{1}{2(k_1 + h^\vee)} \sum_{n \in \mathbb{Z}} : I^{a(1)}_n I^{(1)}_{a,-n} : \qquad T^{(2)}(x) := \frac{1}{2(k_2 + h^\vee)} \sum_{n \in \mathbb{Z}} : I^{a(2)}_n I^{(2)}_{a,-n} :$$

$$T^{(diag)}(x) := \frac{1}{2(k_1 + k_2 + h^\vee)} \sum_{n \in \mathbb{Z}} : (I^{a(1)}_n + I^{a(2)}_n)(I^{(1)}_{a,-n} + I^{(2)}_{a,-n}) :$$

And then the Goddard-Kent-Olive coset generators of Virasoro are:

$$T^{(GKO)}(x) := T^{(1)}(x) + T^{(2)}(x) - T^{(diag)}(x) =: \sum_{n \in \mathbb{Z}} L_n x^{-n-2}$$

Fact: The quadratic Gaudin Hamiltonian is the GKO Virasoro zero mode:

$$\Xi = -(k_1 + k_2 + h^\vee) L_0$$

But the Virasoro algebra is known to have a large commutative subalgebra, called the algebra of **Quantum Integrals of Motion** (of quantum (m)KdV).

[Sasaki, Yamanaka],[Feigin, Frenkel]

$$\begin{aligned} I_1 &= L_0 \\ I_3 &= 2\sum_{n=1}^{\infty} L_{-n}L_n + L_0^2 - \frac{c+2}{12}L_0 + \frac{c(5c+22)}{2880} \\ I_5 &= \dots \end{aligned}$$

Since the first of these is the quadratic Gaudin Hamiltonian, have natural:

Conjecture/Definition: [Feigin, Frenkel] In this case (2 sites, $\widehat{\mathfrak{sl}}_2$) the higher Quantum Integrals of Motion <u>are</u> the higher affine Gaudin Hamiltonians.

Taking this as a definition, have an arena to test conjecture about eigenvalues. . .

$\mathbb{C}$

$\times$ $z_1 = 0$ $\qquad$ $\times$ $z_2 = 1$

$$\boxed{L_{a\Lambda_0 + b\Lambda_1} \otimes L_{\Lambda_0} = L_{(a+1)\Lambda_0 + b\Lambda_1} \otimes \mathcal{U} \oplus \dots}$$

Top multiplicity space = Virasoro module with

$$c(a,b) = 1 - \frac{6}{(a+b+2)(a+b+3)}$$

$$\Delta(a,b) = \frac{b(b+2)}{4(a+b+2)(a+b+3)}$$

- Virasoro calculation: Vacuum value of, e.g. I_5 is

$$I_5 = \Delta^3 - \frac{c+4}{8}\Delta^2 + \frac{(c+2)(3c+20)}{576}\Delta + \frac{(-c)(3c+14)(7c+68)}{290304}$$

- Affine oper calculation: $u(z) := \frac{\frac{1}{4}(b-a)}{z} - \frac{\frac{1}{4}}{z-1}$, $\varphi(z) := \frac{a+b}{z} + \frac{1}{z-1}$ and

$$I_\gamma^{(5)} = \int_\gamma \mathcal{P}(z)^{-5/2} v_5(z) dz$$

where $v_5(z)$ is given by

$$\frac{u(z)^2\left(\frac{d^3}{dz^3}\varphi(z)\right)}{16} + \frac{5u(z)\left(\frac{d}{dz}u(z)\right)\left(\frac{d^2}{dz^2}\varphi(z)\right)}{16} + \frac{-11u(z)^2\varphi(z)\left(\frac{d^2}{dz^2}\varphi(z)\right)}{16} + \frac{-7u(z)^2\left(\frac{d}{dz}\varphi(z)\right)^2}{16} + \frac{5u(z)\left(\frac{d^2}{dz^2}u(z)\right)\left(\frac{d}{dz}\varphi(z)\right)}{8}$$
$$+ \frac{-45u(z)\varphi(z)\left(\frac{d}{dz}u(z)\right)\left(\frac{d}{dz}\varphi(z)\right)}{16} + \frac{23u(z)^2\varphi(z)^2\left(\frac{d}{dz}\varphi(z)\right)}{8} + \frac{-7u(z)^4\left(\frac{d}{dz}\varphi(z)\right)}{16} + \frac{-u(z)\left(\frac{d^4}{dz^4}u(z)\right)}{16} + \frac{5u(z)\varphi(z)\left(\frac{d^3}{dz^3}u(z)\right)}{8} + \frac{-35u(z)\varphi(z)^2\left(\frac{d^2}{dz^2}\cdots\right)}{16}$$
$$+ \frac{7u(z)^3\left(\frac{d^2}{dz^2}u(z)\right)}{16} + \frac{11u(z)^2\left(\frac{d}{dz}u(z)\right)^2}{16} + \frac{25u(z)\varphi(z)^3\left(\frac{d}{dz}u(z)\right)}{8} + \frac{-43u(z)^3\varphi(z)\left(\frac{d}{dz}u(z)\right)}{16} + \frac{-3u(z)^2\varphi(z)^4}{2} + \frac{25u(z)^4\varphi(z)^2}{16} + \frac{-u(z)\cdots}{8}$$

- Result: up to constants independent of the $\mathfrak{sl}_2$ weight b, both I_5 and $I^{(5)}_\gamma$ are equal to

$$\boxed{\begin{array}{c} 425k^6 + 6375k^5 - 2898b^2k^4 - 5796bk^4 + 36287k^4 - 28980b^2k^3 - 57960bk^3 + 97245k^3 \\ +3780b^4k^2 + 15120b^3k^2 - 84042b^2k^2 - 198324bk^2 + 121724k^2 + 18900b^4k + 75600b^3k \\ -57960b^2k - 267120bk + 57120k - 1512b^6 - 9072b^5 + 60480b^3 + 12096b^2 - 120960b \end{array}}$$

- Similar checks works with (up to 2) Bethe roots instead of vacuum.
- Also have Cubic Affine Gaudin Hamiltonian, so can also try $\widehat{\mathfrak{sl}}_3$ case:

$\widehat{\mathfrak{sl}}_3$-Coset construction of W_3 algebra

$\mathbb{C}$

$z_1 = 0 \qquad z_2 = 1$

$$\boxed{L_{a\Lambda_0+b\Lambda_1+c\Lambda_2} \otimes L_{\Lambda_0} = L_{(a+1)\Lambda_0+b\Lambda_1+c\Lambda_2} \otimes \mathcal{U} \oplus \dots}$$

Top multiplicity space = W_3 module
[Bais, Bouwknegt, Surridge, Schoutens]

- On specializing to case of 2 points and $\widehat{\mathfrak{sl}}_M$, find $T = \int_\gamma \mathcal{P}(z)^{-1/3} S_1(z) dz$ and

$$\begin{aligned} W &= \int_\gamma \mathcal{P}(z)^{-2/3} S_2(z) dz \\ &\propto \tfrac{1}{3} t_{abc} I^{a(1)}_{-1} I^{b(1)}_{-1} I^{c(1)}_{-1} |0\rangle^{\boldsymbol{k}} \left(-\tfrac{2}{M}k_2\right)\left(-\tfrac{2}{M}k_2 - 1\right)\left(-\tfrac{2}{M}k_2 - 2\right) \\ &\quad + t_{abc} I^{a(1)}_{-1} I^{b(1)}_{-1} I^{c(2)}_{-1} |0\rangle^{\boldsymbol{k}} \left(-\tfrac{2}{M}k_1 - 2\right)\left(-\tfrac{2}{M}k_2 - 1\right)\left(-\tfrac{2}{M}k_2 - 2\right) \\ &\quad + t_{abc} I^{a(1)}_{-1} I^{b(2)}_{-1} I^{c(2)}_{-1} |0\rangle^{\boldsymbol{k}} \left(-\tfrac{2}{M}k_1 - 1\right)\left(-\tfrac{2}{M}k_1 - 2\right)\left(-\tfrac{2}{M}k_2 - 2\right) \\ &\quad + \tfrac{1}{3} t_{abc} I^{a(2)}_{-1} I^{b(2)}_{-1} I^{c(2)}_{-1} |0\rangle^{\boldsymbol{k}} \left(-\tfrac{2}{M}k_1\right)\left(-\tfrac{2}{M}k_1 - 1\right)\left(-\tfrac{2}{M}k_1 - 2\right) \end{aligned}$$

  are the coset conformal and W vectors.
- Quantum Integrals of Motion $I_1, I_2, \quad I_4, I_5, \quad I_7, I_8, \dots$ are known.
- Check at least vacuum values of I_2, I_4, I_5.

Open questions

- Existence proof (or explicit formula) for the higher Hamiltonians?
- Relation to ODE/IM. [Bazhanov, Lukyanov, Zamolodchikov] [Dorey, Dunning, Tateo] [Masoero, Raimondo, Valeri] ?
  ("dual"?)
- Relation to Integrals of Motion in quantum toroidal algebras? [Feigin, Jimbo, Miwa, Mukhin]